

# Sample path large deviations for super-Brownian motion\*

Alexander Schied

Institut für Mathematik, Humboldt-Universität,

Unter den Linden 6, D-10099 Berlin, Germany

e-mail: schied@mathematik.hu-berlin.de

Revised version: September 18, 1995

## Abstract

We derive two large deviation principles of Freidlin-Wentzell type for rescaled super-Brownian motion. For one of the appearing rate functions an integral representation is given and interpreted as ‘Kakutani-Hellinger energy’. As a tool we develop estimates for the Laplace functionals of (historical) super-Brownian motion and certain maximal inequalities. Also it is shown that the Hölder norm of index  $\alpha < 1/2$  of the process  $t \mapsto \langle f, X_t \rangle$  possesses some finite exponential moments provided the function  $f$  is smooth.

*Mathematics Subject Classification:* 60 J 80, 60 F 10.

---

\*This work was supported in part by the Graduiertenkolleg ”Algebraische, analytische und geometrische Methoden und ihre Wechselwirkung in der modernen Mathematik”, Bonn.

*Running title:* Large deviations for super-Brownian motion.

# 1 Introduction and statement of results

In this paper we study Freidlin-Wentzell type large deviation estimates for certain infinite dimensional diffusions. But before stating our results let us briefly recall the classical facts on small random perturbations of finite dimensional dynamical systems: Suppose  $\xi_t^\varepsilon$  ( $0 \leq t \leq 1$ ) is a  $d$ -dimensional diffusion associated with the generator

$$A_\varepsilon = \sqrt{\varepsilon} \sum_{i,j=1}^d a^{ij}(\cdot) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{i=1}^d b_\varepsilon^i(\cdot) \frac{\partial}{\partial x^i},$$

and starting from  $x \in \mathbb{R}^d$ . Assume that the matrix  $a = (a^{ij})$  is positive definite and that the vector fields  $b_\varepsilon = (b_\varepsilon^i)$  converge uniformly to a vector field  $b$  as  $\varepsilon \downarrow 0$ . Under additional boundedness and regularity assumptions one can show that, as  $\varepsilon \downarrow 0$ , the laws of  $\xi^\varepsilon$  satisfy a large deviation principle on  $C([0, 1] : \mathbb{R}^d)$  with speed  $1/\varepsilon$  and a good rate function  $J_x : C([0, 1] : \mathbb{R}^d) \rightarrow [0, \infty]$ . By this statement we mean the following pair of inequalities: If  $A \subset C([0, 1] : \mathbb{R}^d)$  is closed, then

$$\overline{\lim}_{\varepsilon \downarrow 0} \varepsilon \log P \left[ \xi^\varepsilon \in A \right] \leq - \inf_{w \in A} J_x(w),$$

if, on the other hand,  $U$  is open, then

$$\underline{\lim}_{\varepsilon \downarrow 0} \varepsilon \log P \left[ \xi^\varepsilon \in U \right] \geq - \inf_{w \in U} J_x(w).$$

The expression ‘good rate function’ refers to the lower semicontinuity of  $J_x$  and the compactness of the level sets  $\left\{ \omega \mid J_x(\omega) \leq \alpha \right\}$  ( $\alpha \geq 0$ ). If, for example,  $b \equiv 0$  and  $w(0) = x$ , then one can show that the rate function  $J_x(w)$  coincides with the energy of the path  $w$  with respect to the Riemannian metric on  $\mathbb{R}^d$  that is associated with the inverse  $a^{-1}$  of the coefficient matrix  $a$ . An exposition of these results can be found in the lecture notes [Az] by R. Azencott. For the general theory of large deviations we refer to the book [DZ] by A. Dembo and O. Zeitouni.

The purpose of this paper is to study such large deviation probabilities, when the diffusion  $\xi^\varepsilon$  is replaced by some rescaled *super-Brownian motion*. This process arises as a high-density limit of a system of branching particles, where each particle moves according to a time-scaled Brownian motion  $B_t^\sigma := B_{\sigma t}$  ( $t \geq 0$ ), for some  $\sigma \geq 0$ . Formally, super-Brownian motion is a diffusion  $(X_t)_{t \geq 0}$  taking values in the space  $\mathcal{M}_p^+(\mathbb{R}^d)$  consisting of those positive Radon measures  $\mu$  on  $\mathbb{R}^d$  for which  $\int \phi_p(x) \mu(dx) < \infty$ , where  $\phi_p$  denotes the function  $\phi_p(x) = (1 + |x|^2)^{-p/2}$  and  $p > d$  is fixed. We endow the space  $\mathcal{M}_p^+(\mathbb{R}^d)$  with the topology generated by the maps

$$\mathcal{M}_p^+(\mathbb{R}^d) \ni \mu \longmapsto \langle f, \mu \rangle \quad \left( f \in \{ \phi_p \} \cup C_c(\mathbb{R}^d) \right),$$

with  $C_c(\mathbb{R}^d)$  denoting the space of continuous functions on  $\mathbb{R}^d$  with compact support. The topological space  $\mathcal{M}_p^+(\mathbb{R}^d)$  is Polish, but not locally compact. The law  $\mathbb{P}_\mu^\sigma$  of the process starting from  $\mu$  then is a Borel probability measure on the space  $C([0, \infty) : \mathcal{M}_p^+(\mathbb{R}^d))$  of continuous  $\mathcal{M}_p^+(\mathbb{R}^d)$ -valued path. Throughout this article we will assume that all path spaces are endowed with the

compact open or uniform topology.  $\mathbb{P}_\mu^\sigma$  may be characterized by the Laplace functionals of its transition probabilities:

$$(1) \quad \mathbb{E}_\mu^\sigma \left[ \exp \left( -\langle f, X_t \rangle \right) \right] = \exp \left( -\langle u(t), \mu \rangle \right) \quad \left( f \geq 0, \mu \in \mathcal{M}_p^+(\mathbb{R}^d), t \geq 0 \right).$$

Here  $\langle f, \mu \rangle := \int f d\mu$  as usual, and  $u$  is the unique positive mild solution of the reaction-diffusion equation

$$(2) \quad \begin{cases} \frac{\partial}{\partial t} u &= \frac{\sigma}{2} \Delta u - u^2 \\ u(0) &= f. \end{cases}$$

The reader may find a comprehensive introduction to superprocesses in the lecture notes [Da] by D. Dawson.

One can show that the infinitesimal generator  $L^\sigma$  of super-Brownian motion is, for a large class of ‘smooth’ functions  $F$  on  $\mathcal{M}_p^+(\mathbb{R}^d)$ , given by

$$(3) \quad L^\sigma F(\mu) = \langle F''(\mu), \mu \rangle + \frac{\sigma}{2} \langle \Delta F'(\mu), \mu \rangle \quad (\mu \in \mathcal{M}_p^+(\mathbb{R}^d)),$$

where  $F'_x$  and  $F''_x$  are the Gateaux derivatives of  $F$  in direction of the Dirac measure  $\delta_x$ :

$$F'_x(\mu) = \frac{d}{dt} \Big|_{t=0} F(\mu + t\delta_x) \quad \text{and} \quad F''_x(\mu) = \frac{d^2}{dt^2} \Big|_{t=0} F(\mu + t\delta_x)$$

(cf. [EKR]). Let us now scale, as above, the second order term of  $L^\sigma$ :

$$L_\varepsilon^\sigma F(\mu) := \varepsilon \langle F''(\mu), \mu \rangle + \frac{\sigma}{2} \langle \Delta F'(\mu), \mu \rangle.$$

Then an easy calculation shows that the solution of the corresponding martingale problem coincides with the distribution of  $\varepsilon X$  under  $\mathbb{P}_{\mu/\varepsilon}$ . Now we can state our first result:

**Theorem 1** *Assume  $\sigma \geq 0$  and  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$ . Then, as  $\varepsilon \downarrow 0$ , the distributions of  $(\varepsilon X_t)_{0 \leq t \leq 1}$  with respect to  $\mathbb{P}_{\mu/\varepsilon}^\sigma$  satisfy a large deviation principle with speed  $1/\varepsilon$  and good rate function*

$$(4) \quad I_\mu^\sigma(\omega) = \sup_{f \in C_c([0,1] \times \mathbb{R}^d)} \left( \int_0^1 \langle f(t), \omega(t) \rangle dt - \log \mathbb{E}_\mu^\sigma \left[ \exp \left( \int_0^1 \langle f(t), X_t \rangle dt \right) \right] \right) \\ (\omega \in C([0,1] : \mathcal{M}_p^+(\mathbb{R}^d))).$$

The same problem has been studied independently in the paper [FGK] by K. Fleischmann, J. Gärtner and I. Kaj. In that article the large deviation principle is established by using different methods and only in a weaker topology by embedding  $C([0,1] : \mathcal{M}_p^+(\mathbb{R}^d))$  into a certain locally compact space, while we here deal with the compact open topology. On the other hand, the main emphasis of [FGK] is to find an integral representation for the rate function  $I_\mu^\sigma$ . It is concluded that, for strictly positive  $\sigma$ ,

$$(5) \quad I_\mu^\sigma(\omega) = \begin{cases} \frac{1}{4} \int_0^1 \left\| \frac{d(\dot{\omega}(t) - \frac{\sigma}{2} \Delta \omega(t))}{d\omega(t)} \right\|_{L^2(\omega(t))}^2 dt & \text{if } \omega \in H^\sigma \text{ and } \omega(0) = \mu \\ \infty & \text{else.} \end{cases}$$

Here the Laplace operator is to be taken in sense of Schwartz distributions and the space  $H^\sigma$  consists of all absolutely continuous paths  $\omega \in C\left([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d)\right)$  for which, for almost every  $t$ , the Schwartz distribution  $\dot{\omega}(t) - \frac{\sigma}{2}\Delta\omega(t)$  possesses the Radon-Nikodym derivative  $d(\dot{\omega}(t) - \frac{\sigma}{2}\Delta\omega(t))/d\omega(t)$  with respect to  $\omega(t)$ . However, a complete proof of (5) is only provided if  $\Delta$  is the Laplace operator with Dirichlet boundary conditions for some bounded domain in  $\mathbb{R}^d$ . The extension to the unbounded case, that would be of interest here, relies on some additional hypothesis unproven yet. In Theorem 4 below we will prove an analogue to (5) for the case  $\sigma = 0$ .

Theorem 1 can also be interpreted as an infinite and continuous parameter version of the classical Cramér theorem: Indeed, for  $\varepsilon = 1/n$ , the distribution of  $\frac{1}{n}X$  under  $\mathbb{P}_{n\mu}^\sigma$  coincides with the law of the empirical mean of  $n$  independent super-Brownian motions distributed according to  $\mathbb{P}_\mu^\sigma$ . This follows easily from the branching property (1).

Our next result treats the case where also the drift term of the operator in (3) is scaled, i.e., where the constant  $\sigma$  in Theorem 1 is replaced by a varying scale  $\delta_\varepsilon$  depending on  $\varepsilon$ .

**Theorem 2** *Assume  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$  has full support and  $(\delta_\varepsilon)_{\varepsilon>0}$  decreases to zero as  $\varepsilon \downarrow 0$ . Then, as  $\varepsilon \downarrow 0$ , the distributions of  $(\varepsilon X_t)_{0 \leq t \leq 1}$  with respect to  $\mathbb{P}_{\mu/\varepsilon}^{\delta_\varepsilon}$  satisfy a large deviation principle with speed  $1/\varepsilon$  and good rate function  $I_\mu^0$ .*

Choosing  $\delta_\varepsilon = \sigma\varepsilon$  in this Theorem we get the following

**Corollary 3** *Assume  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$  has full support. Then, as  $\varepsilon \downarrow 0$ , the laws of  $X_{\varepsilon t}$  ( $0 \leq t \leq 1$ ) under  $\mathbb{P}_\mu^\sigma$  satisfy a large deviation principle with speed  $1/\varepsilon$  and good rate function  $I_\mu^0$ .*

This Corollary is a direct analogue to Theorem 6.4 in Chapter V of [Az]. As there, one can use it to study *the small time behaviour* of super-Brownian motion. Below we will see that the rôle of the Riemannian distance in the classical case is now played by the so-called Kakutani-Hellinger distance on  $\mathcal{M}_p^+(\mathbb{R}^d)$ .

Let us consider another application of Theorem 2. To this end, the random measure  $X_t$  will be regarded as a generalized random field on  $\mathbb{R}^d$ . Let us introduce a space-mass transformation by

$$(6) \quad X_t^r := \frac{1}{r^d} X_t \circ \rho_r^{-1} \quad (r > 0),$$

where  $\rho_r : \mathbb{R}^d \rightarrow \mathbb{R}^d$  denotes the space contraction  $\rho_r(x) = x/r$ . Because the Lebesgue measure,  $\lambda$ , is invariant under this transformation (6), it serves as the natural starting point  $X_0$ . Then, as  $r \uparrow \infty$ , the laws of  $(X_t^r)_{0 \leq t \leq 1}$  under  $\mathbb{P}_\lambda^\sigma$  satisfy a large deviation principle with speed  $r^d$  and good rate function  $I_\lambda^0$ . This follows from the fact that the distribution of the process  $(X_t \circ \rho_r^{-1})_{t \geq 0}$  with respect to  $\mathbb{P}_\mu^\sigma$  coincides with  $\mathbb{P}_{\mu \circ \rho_r^{-1}}^{\sigma/r^2}$  (see the proof of Lemma 5.2 in [DF]).

Let us now turn to the integral representation of the rate function  $I_\mu^0$ . Recall that the *Kakutani-Hellinger distance*  $d(\mu, \nu)$  of two positive measures  $\mu$  and  $\nu$  is defined by

$$d(\mu, \nu) = \left( \frac{1}{2} \int \left( \sqrt{\frac{d\mu}{d\eta}} - \sqrt{\frac{d\nu}{d\eta}} \right)^2 d\eta \right)^{1/2},$$

where  $\eta$  is any positive measure such that both  $\mu$  and  $\nu$  are absolutely continuous with respect to  $\eta$  (see [JS]). One can show that  $d(\mu, \nu)$  does not depend upon the particular choice of  $\eta$ . If  $\omega \in C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$  and  $\Delta = \{t_0, \dots, t_n\}$  is any ordered partition of  $[0, 1]$ , we define  $\mathcal{E}_\Delta(\omega)$  by

$$\mathcal{E}_\Delta(\omega) = \sum_{i=1}^n \frac{d(\omega(t_i), \omega(t_{i-1}))^2}{t_i - t_{i-1}}.$$

The expression  $\mathcal{E}(\omega) := \sup_\Delta \mathcal{E}_\Delta(\omega)$  will be called the *Kakutani-Hellinger energy* of the path  $\omega$ . We define  $\mathbf{H}$  to be the set of all those  $\omega \in C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$  that are of the form  $\omega(t) = \omega(0) + \int_0^t \dot{\omega}(s) ds$  ( $0 \leq t \leq 1$ ) for some locally finite signed measures  $\dot{\omega}(s)$ . In addition we demand that  $\dot{\omega}(s)$  be absolutely continuous with respect to  $\omega(s)$  for almost every  $s$  and that

$$\int_0^1 \left\| \frac{d\dot{\omega}(s)}{d\omega(s)} \right\|_{L^2(\omega(s))}^2 ds < \infty.$$

**Theorem 4** *Let  $\mathcal{D}$  denote the set of all  $\omega \in C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$  for which the support  $\text{supp } \omega(t)$  of  $\omega(t)$  is contained in  $\text{supp } \omega(s)$  whenever  $0 \leq s \leq t \leq 1$ . Then*

$$I_\mu^0(\omega) = \begin{cases} 2\mathcal{E}(\omega) & \text{if } \omega \in \mathcal{D} \text{ and } \omega(0) = \mu, \\ \infty & \text{otherwise.} \end{cases}$$

The Kakutani-Hellinger energy  $\mathcal{E}$  satisfies

$$\mathcal{E}(\omega) = \begin{cases} \frac{1}{8} \int_0^1 \left\| \frac{d\dot{\omega}(t)}{d\omega(t)} \right\|_{L^2(\omega(t))}^2 dt & \text{if } \omega \in \mathbf{H}, \\ \infty & \text{otherwise.} \end{cases}$$

It is just for simplicity that we restricted ourselves to the study of super-Brownian motion. Indeed, we could have worked from the beginning with  $\alpha$ -stable processes instead of Brownian motion, as is done in numerous papers on the subject. Also it should be easy to prove the Theorems 1, 2 and 4 also for *historical* super-Brownian motion. If one would like to extend our results to even more general one-particle processes, the key seems to be to find analogues for the results of Section 3. Certainly, the restriction to binary branching is inevitable for large deviations, since otherwise  $\langle f, X_t \rangle$  would not possess any exponential moments for positive  $f$ . However, the results in the sections 5 and 6 also extend to this case with the appropriate modifications.

## 2 Plan of the paper

We start with two preliminary sections on  $p$ -tempered measures and historical super-Brownian motion. Then we state several auxiliary estimates for (historical) super-Brownian motion. Some of them might be interesting in their own:

Some of the major difficulties in the proof of Theorems 1 to 2 stem from the fact that, for every positive function  $f$ , one can find some  $\beta > 0$  such that

$$\mathbb{E}_\mu^\sigma \left[ \exp \left( \beta \langle f, X_t \rangle \right) \right] = +\infty.$$

Using functional analysis, it has been shown in [FK] that there exists a constant  $\varepsilon > 0$  such that at least the expectation  $\mathbb{E}_\mu \left[ \exp \left( \varepsilon \int_0^1 \langle \phi_p, X_s \rangle ds \right) \right]$  becomes finite. This suffices to prove Theorem 1 in a locally compact topology. Its formulation in the strong topology and the statement of Theorem 2 only became possible by the upper bound of the following estimate: If  $f$  is a bounded and measurable function on  $\mathbb{R}^d$  satisfying  $f < 1/t$  a. e., then

$$(7) \quad \frac{E_x \left[ f(B_{\sigma t}) \right]}{1 - t \cdot E_x \left[ f(B_{\sigma t}) \right]} \leq \log \mathbb{E}_{\delta_x}^\sigma \left[ \exp \left( \langle f, X_t \rangle \right) \right] \leq E_x \left[ \frac{f(B_{\sigma t})}{1 - t \cdot f(B_{\sigma t})} \right],$$

where  $E_x$  denotes the expectation with respect to Wiener measure with starting point  $x \in \mathbb{R}^d$ . In Section 5 we prove the corresponding result for historical super-Brownian motion.

With the maximal inequalities of Section 6 one can estimate probabilities of the form  $\mathbb{P}_\mu^\sigma \left[ \sup_{0 \leq t \leq 1} X_t(A) \geq L \right]$ . As an upper bound, one gets expressions involving the first entrance time  $\tau$  of Brownian motion into the set  $A$ . E.g.:

$$\mathbb{P}_\mu^1 \left[ \sup_{0 \leq t \leq 1} X_t(A) \geq L \right] \leq \frac{1}{L} \cdot P_\mu \left[ \tau \leq 1 \right].$$

The results of this Section will be needed to prove exponential tightness and they provide the reason for working with *historical* super-Brownian motion.

Section 7 contains an estimate designed for an application of the exponential tightness criteria in [S1]. Additionally it is shown that the Hölder norm of the stochastic process  $\langle f, X_t \rangle$  possesses some exponential moments with respect to  $\mathbb{P}_\mu^\sigma$  for a large class of smooth functions  $f$ .

The technical Section 8 prepares the application of the Gärtner-Ellis theorem in the proof of Theorem 2. Finally, the proofs of our large deviation results are contained in Section 9.

### 3 $p$ -tempered measures and functions

In this section we state some preliminary results on the  $p$ -tempered measures and functions having been introduced in [Is].

Let  $p > d$  and  $\phi_p(x) = (1 + |x|^2)^{-p/2}$  ( $x \in \mathbb{R}^d$ ). The space  $\mathcal{M}_p^+(\mathbb{R}^d)$  of all  $p$ -tempered measures then consists of all positive Radon measures  $\mu$  on  $\mathbb{R}^d$  that are of the form  $\mu(dx) = \phi_p(x)^{-1} \nu(dx)$  for some positive finite measure  $\nu$  on  $\mathbb{R}^d$ . We endow the space  $\mathcal{M}_p^+(\mathbb{R}^d)$  with the  $p$ -weak topology that is generated by the maps  $\mathcal{M}_p^+(\mathbb{R}^d) \ni \mu \mapsto \langle f, \mu \rangle$  ( $f \in \{\phi_p\} \cup C_c(\mathbb{R}^d)$ ). Then, by definition,  $\mathcal{M}_p^+(\mathbb{R}^d)$  is topologically isomorphic to the space  $M^+(\mathbb{R}^d)$  of all positive finite measures equipped with the topology generated by the maps  $M^+(\mathbb{R}^d) \ni \mu \mapsto \langle f, \mu \rangle$  ( $f \in \{1\} \cup C_c(\mathbb{R}^d)$ ). But this topology coincides with the usual weak topology. As a consequence, we can state the following version of Prohorov's theorem.

**Proposition 5** *A subset  $K \subset \mathcal{M}_p^+(\mathbb{R}^d)$  is relatively compact, if and only if the following two conditions are fulfilled:*

$$(8) \quad \sup_{\mu \in K} \langle \phi_p, \mu \rangle < \infty \quad \text{and} \quad \lim_{R \uparrow \infty} \sup_{\mu \in K} \int_{\{|x| \geq R\}} \phi_p(x) \mu(dx) = 0.$$

Our next lemma is an extension of the branching property (1) to functions taking arbitrary signs.

**Lemma 6** *Assume  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$  and  $f$  is a measurable function on  $\mathbb{R}^d$  satisfying the condition  $E_\mu \left[ |f(B_{\sigma t})| \right] := \int E_x \left[ |f(B_{\sigma t})| \right] \mu(dx) < \infty$ . Then*

$$\log \mathbb{E}_\mu^\sigma \left[ \exp \left( \langle f, X_t \rangle \right) \right] = \int \log \mathbb{E}_{\delta_x}^\sigma \left[ \exp \left( \langle f, X_t \rangle \right) \right] \mu(dx).$$

*Proof:* This statement may be proved like (21) below by making use of the Poisson cluster representation of super-Brownian motion.  $\square$

The assumption  $E_\mu[|f(B_t)|] < \infty$  in the above Lemma is needed to ensure that the random variable  $\langle f, X_t \rangle$  is almost surely well defined and finite. In the sequel we will need an appropriate space of test functions for  $\mathcal{M}_p^+(\mathbb{R}^d)$ , which satisfy this finite moment condition. A convenient choice can be made by using the space  $\mathbf{F}_p$  consisting of all those continuous functions  $f$  for which there exists a finite limit of the ratio  $f(x)/\phi_p(x)$  as  $|x| \uparrow \infty$ . When equipped with the norm

$$(9) \quad \|f\|_{\mathbf{F}_p} = \|f/\phi_p\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)/\phi_p(x)| \quad (f \in \mathbf{F}_p),$$

$\mathbf{F}_p$  becomes isometrically isomorphic to the space of continuous functions on the one-point compactification of  $\mathbb{R}^d$ , endowed with the usual sup-norm. Hence  $\mathbf{F}_p$  is a separable Banach space. By definition,  $\mathcal{M}_p^+(\mathbb{R}^d)$  is a subset of its topological dual and vice versa. Now let us define the semigroup

$$T_t f(x) := E_x[f(B_t)] \quad (t \geq 0, f \in \mathbf{F}_p).$$

Then, according to Lemma (3.2) and Lemma (3.3) of [DFG],  $(T_t)_{t \geq 0}$  is a strongly continuous semigroup of bounded linear operators on  $\mathbf{F}_p$ . In particular, by the uniform boundedness principle, one can find, for each  $t \geq 0$ , a constant  $C_t > 0$  such that

$$(10) \quad \sup_{0 \leq s \leq t} \|T_s f\|_{\mathbf{F}_p} \leq C_t \|f\|_{\mathbf{F}_p} \quad (f \in \mathbf{F}_p).$$

For further reference let us also state a simple consequence of the triangle inequality:

$$(11) \quad f_t \longrightarrow f_0 \text{ in } \mathbf{F}_p \implies T_t f_t \longrightarrow f_0 \text{ in } \mathbf{F}_p \quad (t \downarrow 0).$$

I owe the short proof of the next lemma to Klaus Fleischmann.

**Lemma 7** *Define the function  $\psi_{0,p}$  by*

$$(12) \quad \psi_{0,p}(t, x) = E_x \left[ \sup_{s \leq t} \phi_p(B_s) \right] \quad (t \geq 0, x \in \mathbb{R}^d).$$

*Then  $\psi_{0,p}(t)$  belongs to  $\mathbf{F}_p$  for each  $t \geq 0$ .*

*Proof:* Continuity of  $\psi_{0,p}(t)$  follows from dominated convergence. Now we claim that  $\lim_{|x| \uparrow \infty} \psi_{0,p}(t, x)/\phi_p(x) = 1$ . The lower bound is trivial, since  $\psi_{0,p}(t, x) \geq \phi_p(x)$  for each  $x$ . On the other hand,

$$(13) \quad \frac{\psi_{0,p}(t, x)}{\phi_p(x)} \leq E_0 \left[ \sup_{s \leq t} \frac{\phi_p(x + B_s)}{\phi_p(x)} ; \sup_{s \leq t} |B_s| \leq \delta \right] + \phi_p(x)^{-1} \cdot P_0 \left[ \sup_{s \leq t} |B_s| > \delta \right].$$

Since  $|x + B_s| \geq |x| - |B_s|$ , the first term on the right hand side of (13) may be estimated by  $\phi_p(x)^{-1} \cdot \left(1 + (1 - \delta)^2 |x|^2\right)^{p/2}$ , which converges to  $(1 - \delta)^{-p}$  as  $|x| \uparrow \infty$ . The second term converges to zero, because it follows from the reflection principle for Brownian motion that the probability  $P_0 \left[ \sup_{s \leq t} |B_s| > \delta \right]$  decreases exponentially fast as  $|x| \uparrow \infty$ . Thus we have shown that  $\overline{\lim}_{|x| \uparrow \infty} \psi_{0,p}(t, x)/\phi_p(x) \leq (1 - \delta)^{-p}$ . But  $\delta > 0$  was arbitrary, and hence the assertion follows.  $\square$

## 4 Historical super-Brownian motion

In this section we recall the notion of historical super-Brownian motion and the representation of its Laplace functionals by its canonical measure or Lévy measure. General references for historical superprocesses are [Da] and [Dy]. The definition of a path process below is close to [Dy], section 1.8.

For  $r \geq 0$  and  $\bar{x} \in C([0, \infty) : \mathbb{R}^d)$  we denote by  $\bar{P}_{r, \bar{x}}$  the unique probability measure on  $C([0, \infty) : \mathbb{R}^d)$  such that

$$(14) \quad \text{for } \bar{P}_{r, \bar{x}}\text{-a.e. } \bar{y} \in C([0, \infty) : \mathbb{R}^d), \bar{y}(s) = \bar{x}(s) \text{ for each } s \in [0, r]$$

$$(15) \quad \text{under } \bar{P}_{r, \bar{x}}, \text{ the process } t \mapsto \bar{y}(r+t) \text{ is a Brownian motion starting from } \bar{x}(r).$$

In other words, the measure  $\bar{P}_{r, \bar{x}}$  forces Brownian motion to follow the path  $\bar{x}$  up to time  $r$ . In addition, we consider a *path-valued* stochastic process  $(\bar{B}_t(\bar{y}))_{t \geq 0}$  defined, for  $\bar{y} \in C([0, \infty) : \mathbb{R}^d)$ , by

$$\bar{B}_t(\bar{y}) = \bar{y}(\cdot \wedge t) \quad (t \geq 0).$$

Clearly, the filtration generated by  $(\bar{B}_t)_{t \geq 0}$  coincides with the canonical filtration  $(\mathcal{F}_t)_{t \geq 0}$  that is generated by the coordinate process on  $C([0, \infty) : \mathbb{R}^d)$ . Thus, when  $0 \leq r \leq s$  and  $t \geq 0$ , one easily proves the identity

$$\bar{P}_{r, \bar{x}}[\bar{B}_t \in A | \mathcal{F}_s] = \bar{P}_{s, \bar{B}_s}[\bar{B}_t \in A] \quad \bar{P}_{r, \bar{x}}\text{-a.s.},$$

for cylinder sets  $A$ , and hence for arbitrary Borel sets  $A \subset C([0, \infty) : \mathbb{R}^d)$ . Analogously one can prove a strong Markov property. In this sense the collection  $(\bar{P}_{\cdot, \cdot}, \bar{B}_{\cdot})$  forms a time-inhomogeneous strong Markov process. It is usually called *path process associated with Brownian motion*. We may also consider the case where the underlying Brownian motion is time-scaled by some parameter  $\sigma > 0$ . Then the corresponding probability distribution  $\bar{P}_{r, \bar{x}}^\sigma$  is characterized by assuming, instead of (15), that under  $\bar{P}_{r, \bar{x}}^\sigma$  the process  $t \mapsto \bar{y}(r+t/\sigma)$  behaves like a Brownian motion starting from  $\bar{x}(r)$ .

The superprocess  $\bar{X}$  built upon this path process as one-particle motion is called *historical super-Brownian motion*. It is a time inhomogeneous diffusion possessing as state space the set  $M^+(C([0, \infty) : \mathbb{R}^d))$  of positive finite measures on  $C([0, \infty) : \mathbb{R}^d)$ , and it may be characterized by the Laplace functionals of its transition probabilities:

$$(16) \quad \log \bar{\mathbb{E}}_{r, \bar{\mu}}^\sigma \left[ \exp \left( - \langle f, \bar{X}_t \rangle \right) \right] = - \langle v_t(r), \bar{\mu} \rangle,$$

where  $r \geq 0$ ,  $\bar{\mu} \in M^+(C([0, \infty) : \mathbb{R}^d))$ ,  $f$  denotes a positive bounded and measurable function on  $C([0, \infty) : \mathbb{R}^d)$  and  $v_t$  is the unique positive solution of the non-linear integral equation

$$(17) \quad v_t(r, \bar{x}) = \bar{E}_{r, \bar{x}}^\sigma \left[ f(\bar{B}_t) \right] - \int_r^t \bar{E}_{r, \bar{x}}^\sigma \left[ v_t(s, \bar{B}_s)^2 \right] ds.$$

Thus  $\overline{\mathbb{P}}_{r,\overline{\mu}}^\sigma$  is a probability measure on the space  $C([0, \infty) : M^+(C([0, \infty) : \mathbb{R}^d)))$ . If  $f \equiv \alpha \leq 0$  in (17), then  $v_t$  does not depend on  $\overline{x}$  and is given by  $v_t(r, \overline{x}) = \alpha / (1 + (t - r)\alpha)$ . Analytic extension of this solution to positive  $\alpha$  yields

$$(18) \quad \log \overline{\mathbb{E}}_{r,\overline{\mu}}^\sigma \left[ \exp \left( \alpha \langle 1, \overline{X}_t \rangle \right) \right] = \langle 1, \overline{\mu} \rangle \cdot V_{t-r} \alpha,$$

where

$$(19) \quad V_t \alpha = \begin{cases} \frac{\alpha}{1 - t\alpha} & \text{if } \alpha < \frac{1}{t}, \\ \infty & \text{else} \end{cases} \quad (\alpha \in \mathbb{R}, t \geq 0).$$

For the moment let  $\pi_t$  denote the mapping

$$\pi_t(\overline{x}) = \overline{x}(t) \quad \left( t \geq 0, \overline{x} \in C([0, \infty) : \mathbb{R}^d) \right).$$

If  $r = 0$ , it can be seen immediately that  $\overline{\mathbb{P}}_{0,\overline{\mu}}^\sigma$  only depends on  $\mu := \overline{\mu} \circ \pi_0^{-1}$ . Under slight abuse of notation, we will therefore write  $\overline{\mathbb{P}}_{0,\mu}^\sigma$  instead of  $\overline{\mathbb{P}}_{0,\overline{\mu}}^\sigma$ .

It is not surprising that, if we study the law under  $\overline{\mathbb{P}}_{r,\overline{\mu}}^\sigma$  of the measure-valued process defined by

$$(20) \quad X_t = \overline{X}_{r+t} \circ \pi_{r+t}^{-1} \quad (t \geq 0),$$

we recover ordinary super-Brownian motion starting from  $\overline{\mu} \circ \pi_r^{-1}$ .

**Proposition 8** *For all  $\sigma \geq 0$  and  $r \geq 0$  there exists a unique positive  $\sigma$ -finite measure  $\overline{\Lambda}_r^\sigma$ , concentrated on  $C((r, \infty) : M^+(C([0, \infty) : \mathbb{R}^d)))$ , such that  $\overline{\Lambda}_r^\sigma(\overline{X}_t = 0 \forall t > r) = 0$ , and, for all  $t_1, \dots, t_n > r$  bounded measurable functions  $f_1, \dots, f_n$  on  $C([0, \infty) : \mathbb{R}^d)$  and  $\overline{\mu} \in M^+(C([0, \infty) : \mathbb{R}^d))$*

$$(21) \quad \log \overline{\mathbb{E}}_{r,\overline{\mu}}^\sigma \left[ \exp \left( \sum_{i=1}^n \langle f_i, \overline{X}_{t_i} \rangle \right) \right] = \int \int \exp \left( \sum_{i=1}^n \langle f_i, \delta_{\overline{x}(\cdot \wedge r)} * \overline{X}_{t_i} \rangle \right) - 1 d\overline{\Lambda}_r^\sigma \overline{\mu}(d\overline{x}),$$

where  $*$  denotes convolution. Furthermore,  $\overline{\Lambda}_r^\sigma(\overline{X}_t \neq 0) < \infty$  for every  $t > r$ . We will say that  $\overline{\Lambda}_r^\sigma$  is the **canonical measure** for historical super-Brownian motion.

*Proof:* Uniqueness follows by monotone class arguments.

Let  $\overline{0} \in C([0, \infty) : \mathbb{R}^d)$  denote the trivial path satisfying  $\overline{0}(t) = 0$  for all  $t \geq 0$  and suppose first that, for given  $t_1, \dots, t_n > r$ , the functions  $f_1, \dots, f_n$  only take negative values. Then, by the uniqueness of the solution  $v_t$  of (17) and the Markov property of the historical super-Brownian motion,

$$\overline{\mathbb{E}}_{r,\delta_{\overline{x}}}^\sigma \left[ \exp \left( \sum_{i=1}^n \langle f_i, \overline{X}_{t_i} \rangle \right) \right] = \overline{\mathbb{E}}_{r,\delta_{\overline{0}}}^\sigma \left[ \exp \left( \sum_{i=1}^n \langle f_i, \delta_{\overline{x}(\cdot \wedge r)} * \overline{X}_{t_i} \rangle \right) \right]$$

for all  $\overline{x} \in C([0, \infty) : \mathbb{R}^d)$ . By the branching property (16), we thus may assume  $\overline{\mu} = \delta_{\overline{0}}$ .

To show the existence of  $\bar{\Lambda}_r^\sigma$  one can proceed in different ways. A direct construction using Brownian excursions has been given by J. F. Le Gall (see [LG]). Alternatively, one might use the fact that, by the branching property,  $\bar{\mathbb{P}}_{r, \delta_{\bar{v}}}^\sigma$  defines an infinitely divisible distribution on the semigroup  $C([0, \infty) : M^+(C([0, \infty) : \mathbb{R}^d)))$ . For superprocesses with compact state space, this argument has first been carried out in [EKR]. More generally, one can use ideas of [KMM] to prove a special kind of Lévy-Khinchin representation theorem for infinitely divisible probability measures on semigroups without any topological requirements. This can be applied to general superprocesses. See [S2], sections 1.1 and 1.2.

Now we prove (21) for arbitrary bounded and measurable functions  $f_1, \dots, f_n$  by using the so called Poisson cluster representation. Applying the Markov property, we can reduce the problem to the case where  $n = 1$ . Define the finite measure  $\Gamma$  on the set  $E := C([0, \infty) : \mathbb{R}^d) \times M^+(C([0, \infty) : \mathbb{R}^d))$  by  $\Gamma = \bar{\mu} \otimes \left( \left( \mathbb{1}_{\{\bar{X}_t \neq 0\}} \bar{\Lambda}_r^\sigma \right) \circ \bar{X}_t^{-1} \right)$ . Let  $\Pi_\Gamma$  denote the Poisson measure with intensity  $\Gamma$  and set  $F(\bar{x}, \bar{\nu}) := \langle f, \delta_{\bar{x}(\cdot \wedge r)} * \bar{\nu} \rangle$  ( $(\bar{x}, \bar{\nu}) \in E$ ). Then

$$\int |F| d\Gamma \leq \int \int \langle |f|, \delta_{\bar{x}(\cdot \wedge r)} * \bar{X}_t \rangle d\bar{\Lambda}_r^\sigma \bar{\mu}(d\bar{x}) = \bar{E}_{r, \bar{\mu}}^\sigma \left[ |f(\bar{B}_t)| \right] < \infty.$$

Therefore  $\langle F, \eta \rangle$  is well defined for  $\Pi_\Gamma$ -a.e.  $\eta \in M^+(E)$  and

$$\begin{aligned} \log \int \exp \left( \langle F, \eta \rangle \right) \Pi_\Gamma(d\eta) &= \langle e^F - 1, \Gamma \rangle \\ (22) \qquad \qquad \qquad &= \int \int \exp \left( \langle f, \delta_{\bar{x}(\cdot \wedge r)} * \bar{X}_t \rangle \right) - 1 d\bar{\Lambda}_r^\sigma \bar{\mu}(d\bar{x}). \end{aligned}$$

But since we already know (21) for negative functions, a monotone class argument implies that  $\bar{\mathbb{P}}_{r, \bar{\mu}}^\sigma \circ \bar{X}_t^{-1}$  and  $\Pi_\Gamma \circ Y^{-1}$  coincide, where the random measure  $Y$  is given by  $Y(\eta) = \int \delta_{\bar{x}} * \nu \eta(d\bar{x}, d\nu)$  ( $\eta \in M^+(E)$ ). Hence

$$\bar{\mathbb{E}}_{r, \bar{\nu}}^\sigma \left[ \exp \left( \langle f, \bar{X}_t \rangle \right) \right] = \int \exp \left( \langle f, Y \rangle \right) \Pi_\Gamma(d\eta) = \int \exp \left( \langle F, \eta \rangle \right) \Pi_\Gamma(d\eta)$$

and the assertion now follows from (22). □

*Remark:* Since  $\bar{\Lambda}_r^\sigma$  restricted to the set  $\bigcup_{i=1}^n \{\bar{X}_{t_i} \neq 0\}$  is a finite measure, there arise no difficulties when applying Fubini's theorem to formulas like (21).

## 5 Estimates for the Laplace functionals of historical super-Brownian motion

In this section we give upper and lower bounds for the Laplace functionals of historical super-Brownian motion. The upper bound is crucial for the further development of this paper. As I learned during the preparation of the manuscript, a different though related inequality was first proved in an analytical context by F. B. Weissler (cf. [We], Theorem 3). Using moment estimates,

E. A. Perkins proved Weissler's result for the Laplace functionals of branching particle systems (cf. [Pe], Proposition 2.6).

**Theorem 9** For every bounded and measurable function  $f$  on  $C([0, \infty) : \mathbb{R}^d)$ ,

$$(23) \quad V_{t-r} \overline{E}_{r,\overline{x}}^\sigma [f(\overline{B}_t)] \leq \log \overline{\mathbb{E}}_{r,\delta_{\overline{x}}}^\sigma \left[ \exp \left( \langle f, \overline{X}_t \rangle \right) \right] \leq \overline{E}_{r,\overline{x}}^\sigma [V_{t-r} f(\overline{B}_t)],$$

where  $V_t$  denotes the function defined in (19).

*Remarks:* 1. For bounded and measurable functions  $f$  on  $\mathbb{R}^d$  define the semigroups  $T_t^\sigma$  and  $U_t^\sigma$  ( $t \geq 0$ ) by

$$(24) \quad T_t^\sigma f(x) = E_x [f(B_{\sigma t})] \quad \text{and} \quad U_t^\sigma f(x) = \log \mathbb{E}_{\delta_x}^\sigma \left[ \exp \left( \langle f, X_t \rangle \right) \right].$$

Projecting (23) via (20) yields  $V_t T_t^\sigma f \leq U_t^\sigma f \leq T_t^\sigma V_t f$ , and hence

$$\left( V_{t/n} T_{t/n}^\sigma \right)^n f \leq U_t^\sigma f \leq \left( T_{t/n}^\sigma V_{t/n} \right)^n f.$$

Since  $V_t$  ( $t \geq 0$ ) is a semigroup, too, we may expect the following Trotter product formula:

$$(25) \quad U_t^\sigma = \lim_{n \uparrow \infty} \left( V_{t/n} T_{t/n}^\sigma \right)^n = \lim_{n \uparrow \infty} \left( T_{t/n}^\sigma V_{t/n} \right)^n.$$

In this sense (23) would be optimal. However, stating a rigorous version of (25) would require additional care; in particular, since  $f \mapsto V_t f$  is non-linear and possibly infinite.

2. The results of this section may be extended to general superprocesses, provided the branching parameters are spatially homogeneous. As an example let us consider the case of the so called  $\beta$ -branching, when the square in the defining equation (2) is replaced by the function  $z \mapsto z^{1+\beta}$  for some  $\beta \in (0, 1)$ . Then one would have to replace  $V_t$  by

$$V_t^\beta z = \begin{cases} \frac{z}{(1+t\beta(-z)^\beta)^{1/\beta}} & \text{if } z \leq 0, \\ \infty & \text{otherwise} \end{cases} \quad (z \in \mathbb{R}).$$

3. The upper bound in Theorem 9 can also be extended to the *exit measures* introduced by Dynkin. See [S2].

For our probabilistic proof of Theorem 9 we will need the following Lemma, where we will calculate the conditional expectation of  $\langle f, \overline{X}_t \rangle$  given  $\langle 1, \overline{X}_t \rangle$  with respect to both historical super-Brownian motion  $\overline{\mathbb{P}}_{r,\delta_{\overline{x}}}^\sigma$  and its canonical measure  $\overline{\Lambda}_r^\sigma$ . In the latter case we will use the notation  $\overline{\mathbb{E}}_{r,\delta_{\overline{x}}}^\sigma [\langle f, \overline{X}_t \rangle \mid \langle 1, \overline{X}_t \rangle]$ .

**Lemma 10** Assume  $\sigma \geq 0$ ,  $0 \leq r \leq t < \infty$  and that  $f$  is a bounded and measurable function on  $C([0, \infty) : \mathbb{R}^d)$ . Then

$$1. \quad \overline{\mathbb{E}}_{r,\delta_{\overline{x}}}^\sigma \left[ \langle f, \overline{X}_t \rangle \mid \langle 1, \overline{X}_t \rangle \right] = \overline{E}_{r,\overline{0}}^\sigma [f(\overline{B}_t)] \cdot \langle 1, \overline{X}_t \rangle \quad \overline{\Lambda}_r^\sigma\text{-a.e.}$$

$$2. \quad \overline{\mathbb{E}}_{r,\delta_{\overline{x}}}^\sigma \left[ \langle f, \overline{X}_t \rangle \mid \langle 1, \overline{X}_t \rangle \right] = \overline{E}_{r,\delta_{\overline{x}}}^\sigma [f(\overline{B}_t)] \cdot \langle 1, \overline{X}_t \rangle \quad \overline{\mathbb{P}}_{r,\delta_{\overline{x}}}^\sigma\text{-a.s.}$$

*Proof:* We may assume without restriction that  $f$  is positive. First we will show that for all  $\alpha \geq 0$

$$(26) \quad \int \langle f, \bar{X}_t \rangle \cdot e^{-\alpha \langle 1, \bar{X}_t \rangle} d\bar{\Lambda}_r^\sigma = \bar{E}_{r, \bar{0}}^\sigma [f(\bar{B}_t)] \cdot \int \langle 1, \bar{X}_t \rangle \cdot e^{-\alpha \langle 1, \bar{X}_t \rangle} d\bar{\Lambda}_r^\sigma.$$

This will imply the first assertion. The left hand side of (26) may be calculated as

$$(27) \quad \begin{aligned} & -\frac{d}{ds} \Big|_{s=0} \int \exp \left( -s \langle f, \bar{X}_t \rangle + \alpha \langle 1, \bar{X}_t \rangle \right) - 1 d\bar{\Lambda}_r^\sigma \\ &= \left( \bar{\mathbb{E}}_{r, \bar{0}}^\sigma \left[ \exp \left( -\alpha \langle 1, \bar{X}_t \rangle \right) \right] \right)^{-1} \cdot \bar{\mathbb{E}}_{r, \bar{0}}^\sigma \left[ \langle f, \bar{X}_t \rangle \exp \left( -\alpha \langle 1, \bar{X}_t \rangle \right) \right] \\ &= \bar{E}_{r, \bar{0}}^\sigma \left[ f(\bar{B}_t) \exp \left( -2 \int_r^t v_t(u, \bar{B}_u) du \right) \right], \end{aligned}$$

where  $v_t(r, \bar{x}) := -\log \bar{\mathbb{E}}_{r, \bar{x}}^\sigma \left[ \exp \left( -\alpha \langle 1, \bar{X}_t \rangle \right) \right]$ . The equality (27) can be shown either by the representation formula for Palm distributions of superprocesses (see [Da], Theorem 11.6.1), or directly using a Feynman-Kac argument (see [S2]). But, by (19),  $v_t(r, \bar{x}) = \alpha / (1 + (t-r)\alpha)$  is independent of  $\bar{x}$ , and hence (26) is proved. The second assertion may be reduced to the first by using Proposition 8.  $\square$

**Proof of Theorem 9:** Using Jensen's inequality for conditional expectations and the second assertion of Lemma 10

$$\begin{aligned} \bar{\mathbb{E}}_{r, \delta_{\bar{x}}}^\sigma \left[ \exp \left( \langle f, \bar{X}_t \rangle \right) \right] &\geq \bar{\mathbb{E}}_{r, \delta_{\bar{x}}}^\sigma \left[ \exp \left( \bar{\mathbb{E}}_{r, \delta_{\bar{x}}}^\sigma \left[ \langle f, \bar{X}_t \rangle \mid \langle 1, \bar{X}_t \rangle \right] \right) \right] \\ &= \bar{\mathbb{E}}_{r, \delta_{\bar{x}}}^\sigma \left[ \exp \left( \bar{E}_{r, \bar{x}}^\sigma [f(\bar{B}_t)] \langle 1, \bar{X}_t \rangle \right) \right]. \end{aligned}$$

The lower bound is now implied by (18).

To prove the upper bound, we may assume  $\bar{x} = \bar{0}$ . This time the idea is to apply Jensen's inequality to the integral with respect to the random measure  $\bar{X}_t$ :

$$(28) \quad \begin{aligned} & \log \bar{\mathbb{E}}_{0, \delta_{\bar{0}}}^\sigma \left[ \exp \left( \langle f, \bar{X}_t \rangle \right) \right] \\ &\leq \int_{\{\bar{X}_t \neq 0\}} \frac{1}{\langle 1, \bar{X}_t \rangle} \left\langle e^{\langle 1, \bar{X}_t \rangle \cdot f}, \bar{X}_t \right\rangle - 1 d\bar{\Lambda}_r^\sigma \\ &= \int_{\{\bar{X}_t(\omega) \neq 0\}} \frac{1}{\langle 1, \bar{X}_t(\omega) \rangle} \bar{\mathbb{E}}^{\bar{\Lambda}_r^\sigma} \left[ \left\langle e^{\langle 1, \bar{X}_t(\omega) \rangle \cdot f}, \bar{X}_t \right\rangle \mid \langle 1, \bar{X}_t \rangle \right](\omega) - 1 \bar{\Lambda}_r^\sigma(d\omega). \end{aligned}$$

Lemma 10 applied to (28) gives the upper bound:

$$\log \bar{\mathbb{E}}_{0, \delta_{\bar{0}}}^\sigma \left[ \exp \left( \langle f, \bar{X}_t \rangle \right) \right] \leq \int \bar{E}_{r, \bar{0}}^\sigma \left[ \exp \left( \langle 1, \bar{X}_t \rangle \cdot f(\bar{B}_t) \right) \right] - 1 d\bar{\Lambda}_r^\sigma = \bar{E}_{r, \bar{0}}^\sigma \left[ V_{t-r} f(\bar{B}_t) \right].$$

$\square$

## 6 Maximal inequalities for super-Brownian motion

In this section we present a general method to obtain maximal inequalities for linear functionals of super-Brownian motion. While an inequality of this type first has been derived in the author's diploma thesis, the method of proof presented here is taken from [Br], where it has been applied to sequences  $(\bar{X}_{\tau_n})_{n \in \mathbb{N}}$  of exit measures.

In the sequel we will assume that Brownian motion  $(B_t)_{t \geq 0}$  is represented on its canonical path space  $C([0, \infty) : \mathbb{R}^d)$ ,  $(\mathcal{F}_t)_{t \geq 0}$  is its natural filtration and  $P_\mu^\sigma$  is the scaled Wiener measure with finite initial distribution  $\mu$ . The following lemma is due to Julia Brettschneider.

**Lemma 11** *If  $(M_t)_{t \geq 0}$  is a submartingale with respect to  $P_\mu^\sigma$ , then so is  $(\langle M_t, \bar{X}_t \rangle)_{t \geq 0}$  with respect to  $\bar{\mathbb{P}}_{0, \mu}^\sigma$ .*

*Proof:* If  $(\mathcal{G}_t)_{t \geq 0}$  denotes the natural filtration for historical super-Brownian motion, then, for  $t \geq s$ ,  $\bar{\mathbb{E}}_{0, \mu}^\sigma[\langle M_t, \bar{X}_t \rangle | \mathcal{G}_s] = \bar{\mathbb{E}}_{s, \bar{X}_s}^\sigma[\langle M_t, \bar{X}_t \rangle] = \langle \bar{E}_{s, \cdot}^\sigma[M_t], \bar{X}_s \rangle$ .  $\bar{\mathbb{P}}_{0, \mu}^\sigma$ -a.s. It is easy to see that  $\bar{E}_{s, \cdot}^\sigma[M_t]$  is a (regular) version of the conditional expectation  $E_\mu^\sigma[M_t | \mathcal{F}_s]$ , with  $\mathcal{F}_s := \sigma(B_u : u \leq s)$ . Now define  $C := \left\{ \bar{x} \in C([0, \infty) : \mathbb{R}^d) \mid \bar{E}_{s, \bar{x}}^\sigma[M_t] < M_s(\bar{x}) \right\}$ . Then, since  $C \in \mathcal{F}_s$ ,

$$\bar{\mathbb{E}}_{0, \mu}^\sigma[\bar{X}_s(C)] = \bar{P}_{0, \mu}^\sigma[\bar{B}_s \in C] = P_\mu^\sigma[C] = 0.$$

Thus  $\bar{X}_s(C) = 0$   $\bar{\mathbb{P}}_{0, \mu}^\sigma$ -a.s., and hence

$$\bar{\mathbb{E}}_{0, \mu}^\sigma[\langle M_t, \bar{X}_t \rangle | \mathcal{G}_s] = \langle \bar{E}_{s, \cdot}^\sigma[M_t], \bar{X}_s \rangle \geq \langle M_s, \bar{X}_s \rangle.$$

□

Now let  $f$  denote a positive measurable function on  $\mathbb{R}^d$ . For simplicity we will assume that  $f$  is lower or upper semicontinuous. Then the process  $(M_t)$  given by

$$(29) \quad M_t = \sup_{s \leq t} f(B_s) \quad (t \geq 0)$$

is left or right continuous, respectively, and no measurability problems arise. The idea to obtain inequalities for  $\sup_{s \leq t} \langle f, X_s \rangle$  now is to apply the projection property (20) and Doob's maximal inequalities to the submartingale  $\langle M_t, \bar{X}_t \rangle$  ( $t \geq 0$ ). E.g.

$$(30) \quad \begin{aligned} \mathbb{P}_\mu^\sigma \left[ \sup_{s \leq t} \langle f, X_s \rangle \geq L \right] &\leq \bar{\mathbb{P}}_{0, \mu}^\sigma \left[ \sup_{s \leq t} \langle M_s, \bar{X}_s \rangle \geq L \right] \leq \frac{1}{L} \bar{\mathbb{E}}_{0, \mu}^\sigma \left[ \langle M_t, \bar{X}_t \rangle \right] \\ &= \frac{1}{L} E_\mu^\sigma \left[ \sup_{s \leq t} f(B_s) \right]. \end{aligned}$$

In the sequel we will need the following exponential inequality which also holds for non-finite  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$ . Its proof uses a combination of the above argument and the upper bound of Theorem 9. Therefore it provides the reason why we had to work in the historical setting up to now.

**Proposition 12** Assume  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$ ,  $\sigma \geq 0$  and  $f$  is a lower or upper semicontinuous function taking values in  $[0, 1]$ . Then

$$\mathbb{E}_\mu^\sigma \left[ \exp \left( \frac{1}{2} \sup_{t \leq 1} \langle f, X_t \rangle \right) \right] \leq \exp \left( 1 + \left\langle E. \left[ \sup_{t \leq \sigma} f(B_t) \right], \mu \right\rangle \right).$$

*Proof:* Let  $\mu = \sum_{i=1}^{\infty} \mu_i$  denote a decomposition of the  $\sigma$ -finite measure  $\mu$  into finite measures  $\mu_i$ . Furthermore assume that  $X^1, X^2, \dots$  is a sequence of independent processes such that  $X^i$  has distribution  $\mathbb{P}_{\mu_i}^\sigma$ . Then the distribution of  $\sum_{i=1}^{\infty} X^i$  is equal to  $\mathbb{P}_\mu^\sigma$ . Therefore

$$\sup_{t \leq 1} \langle f, \sum_{i=1}^{\infty} X_t^i \rangle = \sup_{n \in \mathbb{N}} \sup_{t \leq 1} \langle f, \sum_{i=1}^n X_t^i \rangle,$$

and the assumption follows by monotonicity once it is shown for *finite* starting points  $\nu \in M^+(\mathbb{R}^d)$ .

To this end define  $(M_t)_{t \geq 0}$  as in (29). Then, according to Lemma 11, the stochastic process  $\exp \left( \frac{1}{2q} \langle M_t, \bar{X}_t \rangle \right)$  is a submartingale with respect to  $\bar{\mathbb{P}}_{0,\nu}^\sigma$  for each  $q > 0$ . Therefore, arguing similarly to (30) and applying the upper bound of Theorem 9,

$$\begin{aligned} \mathbb{E}_\nu^\sigma \left[ \exp \left( \frac{1}{2} \sup_{t \leq 1} \langle f, X_t \rangle \right) \right] &\leq \bar{\mathbb{E}}_{0,\nu}^\sigma \left[ \left( \sup_{t \leq 1} \exp \left( \frac{1}{2q} \langle M_t, \bar{X}_t \rangle \right) \right)^q \right] \\ &\leq \left( \frac{q}{q-1} \right)^q \cdot \bar{\mathbb{E}}_{0,\nu}^\sigma \left[ \exp \left( \frac{1}{2} \langle M_1, \bar{X}_1 \rangle \right) \right] \\ &\leq \left( \frac{q}{q-1} \right)^q \cdot \exp \left( E_\nu \left[ \sup_{t \leq \sigma} \frac{\frac{1}{2} f(B_t)}{1 - \frac{1}{2} f(B_t)} \right] \right) \\ &\leq \left( \frac{q}{q-1} \right)^q \cdot \exp \left( \left\langle E. \left[ \sup_{t \leq \sigma} f(B_t) \right], \nu \right\rangle \right). \end{aligned}$$

The assertion now follows by letting  $q$  tend to infinity. □

## 7 Exponential moments for the Hölder norm of super-Brownian motion

The aim of this section is to show that, for a large class of functions  $f \in \mathbf{F}_p$ , the stochastic process  $\langle f, X_t \rangle$  is not only  $\mathbb{P}_\mu$ -a.s. Hölder continuous for each Hölder coefficient  $\alpha \in (0, 1/2)$ , but also that the Hölder norm  $|\langle f, X \rangle|_\alpha$  possesses exponential moments with respect to  $\mathbb{P}_\mu^\sigma$ . Hölder continuity of this process has been shown before in [Da], Proposition 7.3.1, but the existence of exponential moments does not follow from that result. J. D. Deuschel and K. Wang proved the existence of exponential moments for certain time integrals of super-Brownian motion. Some of their ideas were used in [S1] to prove an embedding theorem between certain Hölder and Orlicz spaces, that will be applied below.

**Lemma 13** Assume  $\sigma \geq 0$ ,  $s, t \in [0, 1]$ ,  $s \neq t$ ,  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$  and let  $C_\sigma$  denote the constant defined in (10). Suppose furthermore that  $g \in \mathbf{F}_p$  is a twice continuously differentiable function satisfying  $\sigma \Delta g \in \mathbf{F}_p$ ,  $\|g\|_{\mathbf{F}_p} < |t - s|^{-1}$  and

$$(31) \quad \Psi_\sigma(g, |t - s|) := C_\sigma \left( \frac{|t - s| \cdot \|g^2\|_{\mathbf{F}_p}}{1 - |t - s| \cdot \|g\|_{\mathbf{F}_p}} + \frac{\sigma}{2} |t - s| \cdot \|\Delta g\|_{\mathbf{F}_p} \right) < 1.$$

Then

$$(32) \quad \mathbb{E}_\mu^\sigma \left[ \exp \left( \left| \langle g, X_t \rangle - \langle g, X_s \rangle \right| \right) \right] \leq 2 \cdot \exp \left( C_\sigma \langle \phi_p, \mu \rangle \frac{\Psi_\sigma(g, |t - s|)}{1 - \Psi_\sigma(g, |t - s|)} \right).$$

*Proof:* Assume  $t > s$ . Then

$$\begin{aligned} & \mathbb{E}_\mu^\sigma \left[ \exp \left( \left| \langle g, X_t \rangle - \langle g, X_s \rangle \right| \right) \right] \\ & \leq \mathbb{E}_\mu^\sigma \left[ \exp \left( \langle g, X_t \rangle - \langle g, X_s \rangle \right) \right] + \mathbb{E}_\mu^\sigma \left[ \exp \left( \langle g, X_s \rangle - \langle g, X_t \rangle \right) \right]. \end{aligned}$$

Clearly it suffices to estimate only one of the two terms on the right hand side. To this end we apply the Markov property, Lemma 6 and the upper bound of Theorem 9:

$$\begin{aligned} \mathbb{E}_\mu^\sigma \left[ \exp \left( \langle g, X_t \rangle - \langle g, X_s \rangle \right) \right] & \leq \mathbb{E}_\mu^\sigma \left[ \exp \left( \langle T_{\sigma(t-s)} V_{t-s} g - g, X_s \rangle \right) \right] \\ & \leq \exp \left( \langle T_{\sigma s} V_s | T_{\sigma(t-s)} V_{t-s} g - g, \mu \rangle \right) \\ & \leq \exp \left( C_\sigma \langle \phi_p, \mu \rangle \cdot V_1 \|T_{\sigma(t-s)} V_{t-s} g - g\|_{\mathbf{F}_p} \right). \end{aligned}$$

Note that

$$(33) \quad \|T_{\sigma(t-s)} V_{t-s} g - g\|_{\mathbf{F}_p} \leq \|T_{\sigma(t-s)}(V_{t-s} g - g)\|_{\mathbf{F}_p} + \|T_{\sigma(t-s)} g - g\|_{\mathbf{F}_p}.$$

Since  $V_{t-s} \|g\|_{\mathbf{F}_p}$  is finite by assumption, we may estimate the first term on the right hand side of (33) by using (10):

$$\|T_{\sigma(t-s)}(V_{t-s} g - g)\|_{\mathbf{F}_p} \leq C_\sigma \left\| \frac{(t-s)g^2}{1 - (t-s)g} \right\|_{\mathbf{F}_p} \leq C_\sigma \frac{(t-s)\|g^2\|_{\mathbf{F}_p}}{1 - (t-s)\|g\|_{\mathbf{F}_p}}.$$

For the second term in (33) we get

$$\|T_{\sigma(t-s)} g - g\|_{\mathbf{F}_p} = \frac{1}{2} \left\| \int_0^{\sigma(t-s)} T_u \Delta g \, du \right\|_{\mathbf{F}_p} \leq C_\sigma \frac{\sigma(t-s)}{2} \|\Delta g\|_{\mathbf{F}_p}.$$

Putting these estimates together we obtain (32). □

If  $E$  is a vector space with norm  $\|\cdot\|_E$  and  $0 < \alpha < 1$ , we denote by  $H^\alpha([0, 1] : E)$  the space of all continuous  $E$ -valued paths  $w$  with finite Hölder norm

$$|w|_\alpha := \sup_{s \neq t} \frac{\|w(t) - w(s)\|_E}{|t - s|^\alpha}.$$

For  $\Omega := C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$  and  $\kappa > 0$  we also introduce an Orlicz space  $L_{\Phi_\kappa}(\Omega : E, \mathbb{P}_\mu^\sigma)$  with respect to the Young function  $\Phi_\kappa(x) := (e^x - 1)/\kappa$ . It consists of all  $E$ -valued measurable functions  $F$  with finite Luxemburg norm

$$\|F\|_{\Phi_\kappa} := \inf \left\{ \beta > 0 \mid \mathbb{E}_\mu^\sigma \left[ \Phi_\kappa \left( \frac{\|F\|_E}{\beta} \right) \right] \leq 1 \right\}.$$

If  $E = \mathbb{R}$  with the Eukclidean distance, we will just write  $H^\alpha[0, 1]$  and  $L_{\Phi_\kappa}(\Omega, \mathbb{P}_\mu^\sigma)$  respectively. Applying Lemma 13 to the function  $g := \varepsilon|t - s|^{-1/2} \cdot f$ , where  $f \in \mathbf{F}_p$  is twice continuously differentiable with  $\sigma\Delta f \in \mathbf{F}_p$  and  $\varepsilon$  is a suitable constant, we find that

$$\langle f, X_\cdot \rangle \in H^{1/2}([0, 1] : L_{\Phi_2}(\Omega, \mathbb{P}_\mu^\sigma)).$$

But Theorem 2 of [S1] now asserts that, for every  $\alpha \in (0, 1/2)$ ,  $H^{1/2}([0, 1] : L_{\Phi_2}(\Omega, \mathbb{P}_\mu^\sigma))$  is continuously embedded into  $L_{\Phi_2}(\Omega : H^\alpha[0, 1], \mathbb{P}_\mu^\sigma)$ . Thus we have proved:

**Corollary 14** *Assume  $\alpha \in (0, 1/2)$  and  $f \in \mathbf{F}_p$  is twice continuously differentiable such that  $\sigma\Delta f \in \mathbf{F}_p$ . Then the stochastic process  $\langle f, X_\cdot \rangle$  is  $\mathbb{P}_\mu^\sigma$ -a.s. Hölder continuous with exponent  $\alpha$ . Moreover there exists a constant  $\delta > 0$  such that  $\mathbb{E}_\mu^\sigma \left[ \exp \left( \delta |\langle f, X_\cdot \rangle|_\alpha \right) \right] < \infty$ .*

## 8 Convergence of the finite dimensional marginals

In this section  $\mu$  denotes a fixed measure in  $\mathcal{M}_p^+(\mathbb{R}^d)$  having full support in  $\mathbb{R}^d$ . For any natural number  $n$  and all  $0 \leq t_1 < t_2 < \dots < t_n < \infty$ , we define

$$\mathbf{D}_{t_1 \dots t_n} := \left\{ (f_1, \dots, f_n) \in \mathbf{F}_p^n \mid \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=1}^n \langle f_i, X_{t_i} \rangle \right) \right] < \infty \right\}.$$

The interior of  $\mathbf{D}_{t_1 \dots t_n}$  in  $\mathbf{F}_p^n$  will be denoted by  $\overset{\circ}{\mathbf{D}}_{t_1 \dots t_n}$ . In the sequel we will be interested in the convergence of the finite dimensional marginals of super-Brownian motion  $\mathbb{P}_\mu^\delta$  as  $\delta \downarrow 0$ .

**Proposition 15** *For any  $(f_1, \dots, f_n) \in \mathbf{F}_p^n$ ,*

$$(34) \quad \liminf_{\delta \downarrow 0} \mathbb{E}_\mu^\delta \left[ \exp \left( \sum_{i=1}^n \langle f_i, X_{t_i} \rangle \right) \right] \geq \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=1}^n \langle f_i, X_{t_i} \rangle \right) \right].$$

*If additionally  $(f_1, \dots, f_n) \in \overset{\circ}{\mathbf{D}}_{t_1 \dots t_n}$ , then*

$$(35) \quad \mathbb{E}_\mu^\delta \left[ \exp \left( \sum_{i=1}^n \langle f_i, X_{t_i} \rangle \right) \right] \longrightarrow \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=1}^n \langle f_i, X_{t_i} \rangle \right) \right] \quad (\delta \downarrow 0).$$

The proof of this proposition is divided into several lemmas. Assertion (34) will be proved in the sequel to Lemma 16, (35) follows from Lemma 18.  $\mathbf{F}_p^+$  will denote the cone of all positive functions in  $\mathbf{F}_p$ .

**Lemma 16** *Assume the functions  $f_1^\delta, \dots, f_n^\delta \in \mathbf{F}_p^+$  satisfy  $f_i^\delta \rightarrow f_i^0$  in  $\mathbf{F}_p$  ( $\delta \downarrow 0$ ) and  $\sup_{i,x,\delta} f_i^\delta(x) \leq \text{const}$ . If  $u_\delta$  is defined by  $u_\delta(x) = -\log \mathbb{E}_{\delta_x}^\delta \left[ \exp \left( -\sum_{i=1}^n \langle f_i^\delta, X_{t_i} \rangle \right) \right]$ , then  $u_\delta \in \mathbf{F}_p^+$  and*

$$(36) \quad u_\delta \rightarrow u_0 \text{ in } \mathbf{F}_p \text{ as } \delta \downarrow 0.$$

*Proof:*  $u_\delta \in \mathbf{F}_p^+$  follows inductively from [DF], Lemma 2.3.

Now suppose  $n = 1$ . Then (36) is immediate from the upper and lower bounds of Theorem 9 together with (11). In the general case we proceed by induction. Applying the Markov property yields

$$\mathbb{E}_{\delta_x}^\delta \left[ \exp \left( -\sum_{i=1}^{n+1} \langle f_i^\delta, X_{t_i} \rangle \right) \right] = \mathbb{E}_{\delta_x}^\delta \left[ \exp \left( -\sum_{i=1}^{n-1} \langle f_i^\delta, X_{t_i} \rangle - \langle \tilde{f}_n^\delta, X_{t_n} \rangle \right) \right],$$

where  $\tilde{f}_n^\delta(x) := f_n^\delta(x) - \log \mathbb{E}_{\delta_x}^\delta \left[ \exp \left( -\langle f_{n+1}^\delta, X_{t_{n+1}-t_n} \rangle \right) \right]$ . But  $\tilde{f}_n^\delta \in \mathbf{F}_p^+$  and  $\tilde{f}_n^\delta \rightarrow \tilde{f}_n^0$  in  $\mathbf{F}_p$  have already been shown. Therefore the lemma is proved.  $\square$

*Proof of (34):* We may conclude from Lemma 16 that the finite dimensional marginals of  $\mathbb{P}_\mu^\delta$  converge weakly to those of  $\mathbb{P}_\mu^0$ : Define the polish space  $E = \{1, \dots, n\} \times \mathbb{R}^d$  and the finite random measure  $Y$  on  $E$  by  $Y(dk, dx) = \sum_{i=1}^n \delta_i(dk) \phi_p(x) X_{t_i}(dx)$ . Dually, we regard elements of  $(\mathbf{F}_p^+)^n$  as continuous functions on  $E$ . Now we are in the situation of Theorem 3.2.6 of [Da], which asserts the weak convergence.

Having established this, (34) follows by noting that the function

$$Q \mapsto \int \exp \left( \sum_{i=1}^n \langle f_i, \nu_i \rangle \right) Q(d\nu_1, \dots, d\nu_n)$$

is lower semicontinuous for the weak topology on the space of probability measures on  $(\mathcal{M}_p^+(\mathbb{R}^d))^n$ .  $\square$

For any  $a \geq 0$ , we define  $M_a := \left\{ f \in \mathbf{F}_p \mid \max_{x \in \mathbb{R}^d} f(x) \leq a \right\}$ .

**Lemma 17** *Assume  $0 \leq a < 1/t$ .*

1. *If  $f \in M_a$  and  $V_t$  is given by (19), then  $V_t f \in \mathbf{F}_p$  and  $\|V_t f\|_{\mathbf{F}_p} \leq \|f\|_{\mathbf{F}_p} / (1 - ta)$ .*
2. *For  $f, g \in M_a$ ,  $\|V_t f - V_t g\|_{\mathbf{F}_p} \leq \|f - g\|_{\mathbf{F}_p} / (1 - ta)^2$ .*
3.  $\overset{\circ}{\mathbf{D}}_t = \bigcup_{0 \leq a < 1/t} M_a$ . *In particular, the topological boundary  $\partial \mathbf{D}_t$  of  $\mathbf{D}_t$  is given by*

$$\partial \mathbf{D}_t = \left\{ f \in \mathbf{F}_p \mid \max_{x \in \mathbb{R}^d} f(x) = 1/t \right\}.$$

4. Suppose  $(g_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathbf{F}_p$  satisfying  $\|g_n - g\|_{\mathbf{F}_p} \rightarrow 0$  for some  $g \in \mathring{\mathbf{D}}_t$ . Then also  $\|V_t g_n - V_t g\|_{\mathbf{F}_p} \rightarrow 0$  ( $n \uparrow \infty$ ).

*Proof:* 1. If  $f(x) \geq 0$ , then  $0 \leq V_t f(x) \leq f(x)/(1 - at)$ . On the other hand,  $f(x) \leq V_t f(x) \leq 0$ , when  $f(x) \leq 0$ . This proves the first statement.

2. is implied by the identity  $V_t f - V_t g = (f - g)/(1 - tf)(1 - tg)$ .

3. For  $a < 1/t$ ,  $M_a \subset \mathring{\mathbf{D}}_t$  by 1. Now suppose  $f \in \mathring{\mathbf{D}}_t$ . Then we may find an  $\varepsilon > 0$  such that  $\infty > \mathbb{E}_\mu^0 \left[ \exp \left( \langle f + \varepsilon \phi_p, X_t \rangle \right) \right] = \exp \left( \langle V_t(f + \varepsilon \phi_p), \mu \rangle \right)$ . Thus  $f + \varepsilon \phi_p < 1/t$   $\mu$ -a.e., and hence  $\sup_x (f(x) + \varepsilon \phi_p(x)) \leq 1/t$ , since  $\mu$  has full support. But a function in  $\mathbf{F}_p$  takes on its maximum when not  $f \leq 0$ . This shows that  $\max_x f(x) < 1/t$ .

4.  $g \in M_a$  for some  $a < 1/t$  by 3. Since convergence in  $\mathbf{F}_p$  implies uniform convergence,  $g_n \in M_{a+\varepsilon}$  for some  $a < 1/t - \varepsilon$  and for all  $n \geq$  some  $n_0$ . The assertion now follows by an application of 2.  $\square$

**Lemma 18** Assume the functions  $f_1^\delta, \dots, f_n^\delta \in \mathbf{F}_p$  satisfy  $\|f_i^\delta - f_i^0\|_{\mathbf{F}_p} \rightarrow 0$  ( $\delta \downarrow 0$ ,  $i = 1, \dots, n$ ) for some  $(f_1^0, \dots, f_n^0) \in \mathring{\mathbf{D}}_{t_1 \dots t_n}$ . Then

$$\overline{\lim}_{\delta \downarrow 0} \mathbb{E}_\mu^\delta \left[ \exp \left( \sum_{i=1}^n \langle f_i^\delta, X_{t_i} \rangle \right) \right] \leq \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=1}^n \langle f_i^0, X_{t_i} \rangle \right) \right].$$

*Proof:* Without restriction we may assume  $t_1 = 0$ . Then the assertion is trivial for  $n = 1$ .

In the general case we proceed by induction. To keep formulas short, we define  $r := t_{n+1} - t_n$ . The Markov property, Theorem 9 and (10) yield

$$\mathbb{E}_\mu^\delta \left[ \exp \left( \sum_{i=1}^{n+1} \langle f_i^\delta, X_{t_i} \rangle \right) \right] \leq \mathbb{E}_\mu^\delta \left[ \exp \left( \sum_{i=1}^{n-1} \langle f_i^\delta, X_{t_i} \rangle + \langle \tilde{f}_n^\delta, X_{t_n} \rangle \right) \right],$$

where  $\tilde{f}_n^\delta := f_n^\delta + T_{\delta r} V_r f_{n+1}^\delta$  ( $0 \leq \delta \leq 1$ ). Then the Markov property of  $\mathbb{P}_\mu^0$  immediately implies that  $(f_1^0, \dots, f_{n-1}^0, \tilde{f}_n^0) \in \mathring{\mathbf{D}}_{t_1 \dots t_n}$ . If we could show that

$$(37) \quad \tilde{f}_n^\delta \rightarrow \tilde{f}_n^0 = f_n^0 + V_r f_{n+1}^0 \quad \text{in } \mathbf{F}_p \text{ as } \delta \downarrow 0,$$

then the assertion would follow by induction and an application of the Markov property.

To show (37), we will first prove that  $f_{n+1}^0 \in \mathring{\mathbf{D}}_r$ . Indeed, by assumption, there exists an  $\varepsilon > 0$  such that

$$\begin{aligned} \infty &> \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=1}^{n+1} \langle f_i^0, X_{t_i} \rangle + \langle \varepsilon \phi_p, X_{t_{n+1}} \rangle \right) \right] \\ &= \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=1}^n \langle f_i^0, X_{t_i} \rangle + \langle V_r(f_{n+1}^0 + \varepsilon \phi_p), X_{t_n} \rangle \right) \right]. \end{aligned}$$

The set  $C$  defined by  $C := \left\{ x \in \mathbb{R}^d \mid V_r(f_{n+1}^0 + \varepsilon \phi_p)(x) = \infty \right\}$  therefore satisfies  $X_{t_n}(C) = 0$   $\mathbb{P}_\mu^0$ -a.s. Thus  $0 = \mathbb{E}_\mu^0 [X_{t_n}(C)] = \mu(C)$ , and  $f_{n+1}^0 \in \mathring{\mathbf{D}}_r$  follows as in the proof of the third part of Lemma 17.

Now let us prove (37). In view of what we have just shown, part four of Lemma 17 implies that  $V_r f_{n+1}^\delta \rightarrow V_r f_{n+1}^0 = \tilde{f}_n^0 - f_n^0$  in  $\mathbf{F}_p$  as  $\delta \downarrow 0$ . (37) now follows by (11).  $\square$

## 9 Proof of the main results

### 9.1 Proof of Theorem 1:

By the general Cramér theorem (cf. e.g. [DZ], Theorem 6.1.3), at least a weak large deviation principle must hold as  $\varepsilon \downarrow 0$ . Here it is not a restriction that our parameter  $\varepsilon$  is allowed to vary continuously: see [FK] and [FGK]. Thus it suffices to show exponential tightness along each sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  with  $\varepsilon_n \downarrow 0$  ( $n \uparrow \infty$ ) (cf. [DZ], Lemma 1.2.18 (a)). To this end, we will apply the criteria of [S1].

As a first step, we define the functions

$$\psi_{R,p}(t, x) := E_x \left[ \sup_{s \leq t} \mathbf{I}_{U_R^c}(B_s) \cdot \phi_p(B_s) \right] \quad (x \in \mathbb{R}^d, t \geq 0, R \geq 0),$$

where  $U_R = \{x \in \mathbb{R}^d \mid |x| < R\}$ . By Lemma 7 and dominated convergence  $\langle \psi_{R,p}(\sigma), \mu \rangle \downarrow 0$  as  $R \uparrow \infty$ . Hence there exists a sequence  $(R_n)_{n \in \mathbb{N}}$  increasing to infinity such that  $\langle \psi_{R_n,p}(\sigma), \mu \rangle \leq 1/n^2$  and  $(1 + R_n^2)^{-p/2} \leq 1/n^2$  for all  $n$ . Then, for each  $L > 0$ , the set

$$A_L := \left\{ \nu \in \mathcal{M}_p^+(\mathbb{R}^d) \mid \langle \phi_p, \nu \rangle \leq L, \int_{U_{R_n}^c} \phi_p(x) \nu(dx) \leq \frac{L}{n} \quad (n \in \mathbb{N}) \right\}$$

is relatively compact in  $\mathcal{M}_p^+(\mathbb{R}^d)$  by Proposition 5. Applying the exponential Tschebychev inequality, Proposition 12 and Lemma 7 we get

$$\begin{aligned} & \mathbb{P}_{\mu/\varepsilon}^\sigma \left[ \exists t \in [0, 1] : \varepsilon X_t \notin A_L \right] \\ & \leq \mathbb{P}_{\mu/\varepsilon}^\sigma \left[ \sup_{t \leq 1} \langle \phi_p, X_t \rangle \geq L/\varepsilon \right] + \sum_{n=1}^{\infty} \mathbb{P}_{\mu/\varepsilon}^\sigma \left[ \sup_{t \leq 1} \langle n^2 \mathbf{I}_{U_{R_n}^c} \phi_p, X_t \rangle \geq nL/\varepsilon \right] \\ & \leq \exp \left( 1 + \frac{1}{\varepsilon} \langle \psi_{0,p}(\sigma), \mu \rangle - \frac{L}{2\varepsilon} \right) + \sum_{n=1}^{\infty} \exp \left( 1 + \frac{n^2}{\varepsilon} \langle \psi_{R_n,p}(\sigma), \mu \rangle - \frac{nL}{2\varepsilon} \right) \\ & \leq \exp \left( 1 + \frac{1}{\varepsilon} \langle \psi_{0,p}(\sigma), \mu \rangle - \frac{L}{2\varepsilon} \right) + e^{1+1/\varepsilon} \cdot \frac{e^{-L/2\varepsilon}}{1 - e^{-L/2\varepsilon}}. \end{aligned}$$

Consequently,

$$(38) \quad \overline{\lim}_{\varepsilon_n \downarrow 0} \varepsilon_n \log \mathbb{P}_{\mu/\varepsilon_n}^\sigma \left[ \exists t \in [0, 1] : \varepsilon_n X_t \notin A_L \right] \leq 1 \vee \langle \psi_{0,p}(\sigma), \mu \rangle - \frac{L}{2}.$$

But this is just condition (i) of Theorem 1 in [S1].

It follows from Corollary 14 that the distributions of  $\varepsilon_n \langle f, X_\cdot \rangle$  under  $\mathbb{P}_{\mu/\varepsilon_n}^\sigma$  on  $C[0, 1]$  are exponentially tight, provided  $f \in \mathbf{F}_p$  is twice continuously differentiable and  $\sigma \Delta f \in \mathbf{F}_p$  — the

argument is the same as in the proof of Theorem 3 in [S1]. Therefore condition (ii) of Theorem 1 of [S1] is fulfilled and exponential tightness is proved, if we choose

$$(39) \quad \mathbb{F} := \left\{ \langle f, \cdot \rangle \mid f \in \mathbf{F}_p \text{ is twice continuously differentiable and } \sigma \Delta f \in \mathbf{F}_p \right\}.$$

Up to now, we have shown that a full large deviation principle holds with some good rate function  $J$  (cf. [DZ], Lemma 1.2.18). According to Theorem 6.1.3 of [DZ],  $J$  is given as Legendre transform involving the topological dual of  $C\left([0, 1] : \mathcal{M}_p(\mathbb{R}^d)\right)$ , where  $\mathcal{M}_p(\mathbb{R}^d) := \mathcal{M}_p^+(\mathbb{R}^d) - \mathcal{M}_p^+(\mathbb{R}^d)$ . To identify it with  $I_\mu^\sigma$ , simply embed  $C\left([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d)\right)$  into the space  $\mathcal{M}^+\left([0, 1] \times \mathbb{R}^d\right)$  of all positive Radon measures on  $[0, 1] \times \mathbb{R}^d$  by setting  $\langle \psi, \omega \rangle := \int_0^1 \langle \psi(t), \omega(t) \rangle dt$ , for  $\omega \in C\left([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d)\right)$  and  $\psi \in C_c\left([0, 1] \times \mathbb{R}^d\right)$ . We endow  $\mathcal{M}^+\left([0, 1] \times \mathbb{R}^d\right)$  with the vague topology and may apply Theorem 6.1.3 of [DZ] again, to find a large deviation principle with rate function  $I_\mu^\sigma$ . But the above embedding is injective and thus  $J$  and  $I_\mu^\sigma$  must coincide by the contraction principle and the uniqueness of the rate function (cf. [DZ], Theorem 4.2.1 and Lemma 4.1.4, respectively). This proves Theorem 1.  $\square$

## 9.2 Proof of Theorem 2:

We prove Theorem 2 for a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$ , and we let  $\delta_n := \delta_{\varepsilon_n}$ . First we show exponential tightness in the space  $C\left([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d)\right)$  endowed with the compact-open topology. Then, automatically, any auxiliary large deviation principle considered during the proof will be exponentially tight, too. Again we will use Theorem 1 of [S1]. That its condition (i) is fulfilled can be seen exactly as in the proof of Theorem 1. Also the set  $\mathbb{F}$  will be chosen like in (39). Then, if  $f \in \mathbb{F}$  and  $\gamma > 0$  is small enough, Lemma 13 yields for large  $n$

$$\mathbb{E}_{\mu/\varepsilon_n}^{\delta_n} \left[ \exp \left( \frac{\gamma}{\varepsilon_n \sqrt{|t-s|}} \left| \langle f, \varepsilon_n X_t \rangle - \langle f, \varepsilon_n X_s \rangle \right| \right) \right] \leq \kappa^{1/\varepsilon_n} \quad (s \neq t \in [0, 1]),$$

where  $\kappa = 2 \exp\left(C_1 \langle \phi_p, \mu \rangle V_1 \Psi_1(\gamma f, 1)\right)$ . Exponential tightness now follows from Theorem 3 of [S1].

Now we formulate a general argument relying on Dawson's and Gärtner's theorem for projective systems and large deviations. We refer to [DZ], Section 4.6, for the notion of a projective system.

**Lemma 19** *Assume  $(\mathcal{Y}_j, p_{ij})_{i \leq j \in J}$  is a projective system,  $E$  is a topological Hausdorff space and  $(q_j)_{j \in J}$  is a family of mappings separating the points of  $E$  such that  $q_j : E \rightarrow \mathcal{Y}_j$  and  $q_i = p_{ij} \circ q_j$  ( $i \leq j \in J$ ). If  $(\mu_n)_{n \in \mathbb{N}}$  is an exponential tight family of Borel probability measures on  $E$ , such that, for every  $j \in J$ ,  $(\mu_n \circ q_j^{-1})_{n \in \mathbb{N}}$  satisfies a large deviation principle with speed  $(1/\varepsilon_n)_{n \in \mathbb{N}}$  and rate function  $I_j$ , then  $(\mu_n)_{n \in \mathbb{N}}$  satisfies a large deviation principle with speed  $(1/\varepsilon_n)_{n \in \mathbb{N}}$  and good rate function  $I(x) = \sup_{j \in J} I_j(q_j(x))$ .*

*Proof:* By our assumptions on the family  $(q_j)_{j \in J}$ ,  $E$  may be continuously embedded into the projective limit  $\mathcal{X} = \varprojlim \mathcal{Y}_j$ . Furthermore note that each  $I_j$  is a good rate function on  $\mathcal{Y}_j$  by Lemma 1.2.18 (b) of [DZ]. By the Dawson-Gärtner theorem (cf. [DZ], Theorem 4.6.1),  $(\mu_n)_{n \in \mathbb{N}}$  satisfies in  $\mathcal{X}$  a large deviation principle with good rate function  $\tilde{I}(\tilde{x}) = \sup_{j \in J} I_j(p_j(\tilde{x}))$  ( $\tilde{x} \in \mathcal{X}$ ), where  $p_j$  denotes the canonical map  $p_j : \mathcal{X} \rightarrow \mathcal{Y}_j$  ( $j \in J$ ). But exponential tightness in  $E$  implies  $\tilde{I}(\tilde{x}) = \infty$  whenever  $\tilde{x} \in \mathcal{X} - E$ . The assertion now follows from Lemma 4.1.5 (b) and Corollary 4.2.6 of [DZ].  $\square$

By the aid of this lemma, we may reduce our original problem to the large deviations of the finite dimensional marginals. Indeed, choose  $J$  to be the set of all finite subsets  $j \subset [0, 1]$  partially ordered by inclusion,  $\mathcal{Y}_j = (\mathcal{M}_p^+(\mathbb{R}^d))^j$ ,  $E = C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$  and define  $q_j$  by  $q_j(\omega)(t) = \omega(t)$  ( $\omega \in E, t \in j \in J$ ). The next lemma shows which rate function  $I_j$  we should expect to govern the large deviations in  $\mathcal{Y}_j$ .

**Lemma 20** *Suppose  $I_j(\vec{\nu})$  is, for  $\vec{\nu} \in (\mathcal{M}_p^+(\mathbb{R}^d))^j$ , given by*

$$I_j(\vec{\nu}) = \sup_{\vec{f} \in (C_c(\mathbb{R}^d))^j} \left( \sum_{t \in j} \langle \vec{f}(t), \vec{\nu}(t) \rangle - \log \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{t \in j} \langle \vec{f}(t), X_t \rangle \right) \right] \right).$$

*Then  $I_\mu^0(\omega) = \sup_{j \in J} I_j(q_j(\omega))$ , for each  $\omega \in C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$ .*

*Proof:* Apply the general Cramér theorem (Theorem 6.1.3 of [DZ]) to the distribution of  $q_j(\frac{1}{n}X)$  with respect to  $\mathbb{P}_{n\mu}^0$  ( $n \in \mathbb{N}, j \in J$ ). When  $\mathcal{M}_p^+(\mathbb{R}^d)$  is endowed with the vague topology, we get a large deviation principle with state space  $(\mathcal{M}_p^+(\mathbb{R}^d))^j$  and rate function  $I_j$ , since exponential tightness holds by the proof of Theorem 1. Lemma 19 now yields a large deviation principle in  $C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$  with rate function  $\sup_{j \in J} I_j(q_j(\omega))$ . But the same large deviation principle follows from Theorem 1. Thus  $I_\mu^0 = \sup_{j \in J} I_j(q_j(\omega))$ , since rate functions are unique (Lemma 4.1.4 of [DZ]).  $\square$

The next step is to prove a large deviation principle for the  $(\mathcal{M}_p^+(\mathbb{R}^d))^k$ -valued random variables  $\varepsilon_n(X_{t_1}, \dots, X_{t_k})$ , where  $k \in \mathbb{N}$  and  $0 \leq t_1 < \dots < t_k \leq 1$  are fixed. For brevity, define the functional  $Z$  on  $(C_c(\mathbb{R}^d))^k$  by

$$Z(f_1, \dots, f_k) = \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=1}^k \langle f_i, X_{t_i} \rangle \right) \right] \quad (f_1, \dots, f_k \in C_c(\mathbb{R}^d)).$$

Now suppose that  $H$  is a finite dimensional subspace of  $C_c(\mathbb{R}^d)$  satisfying

$$(40) \quad Z(f_1, \dots, f_k) = \infty \quad \text{whenever } (f_1, \dots, f_k) \in H^k \cap \partial \mathbf{D}_{t_1 \dots t_k},$$

where  $\partial \mathbf{D}_{t_1 \dots t_k}$  denotes the topological boundary in  $\mathbf{F}_p^k$  of the set  $\mathbf{D}_{t_1 \dots t_k}$  defined in Section 8. Then it follows immediately from Lemma 6 and Proposition 15 that, for  $(f_1, \dots, f_k) \in H^k$ , as  $n \uparrow \infty$ ,

$$(41) \quad \varepsilon_n \log \mathbb{E}_{\mu/\varepsilon_n}^{\delta_n} \left[ \exp \left( \sum_{i=1}^k \langle f_i, X_{t_i} \rangle \right) \right] \rightarrow \log Z(f_1, \dots, f_k).$$

**Lemma 21** Under assumption (40), the convex function  $\log Z : H^k \rightarrow (-\infty, \infty]$  is lower semi-continuous and essentially smooth. I.e.

1. The set  $\mathbf{D} := H^k \cap \mathbf{D}_{t_1 \dots t_k}$  possesses nonempty interior  $\overset{\circ}{\mathbf{D}}$ ,
2.  $\log Z$  is differentiable throughout  $\overset{\circ}{\mathbf{D}}$ ,
3.  $\log Z$  is steep, namely  $\lim_{n \uparrow \infty} |\nabla \log Z(\vec{f}_n)| = \infty$  whenever  $(\vec{f}_n)_{n \in \mathbb{N}}$  is a sequence in  $\overset{\circ}{\mathbf{D}}$  converging to a boundary point of  $\mathbf{D}$ .

*Proof:* 1. is obvious and 2. is a standard application of Lebesgue's dominated convergence theorem. Lower semicontinuity follows from Fatou's lemma. Together with assumption (40) it implies  $\lim_{\vec{f} \rightarrow \partial \mathbf{D}} \log Z_k(\vec{f}) = \infty$ , which yields 3.  $\square$

Every  $\vec{\nu} = (\nu_1, \dots, \nu_k) \in (\mathcal{M}_p^+(\mathbb{R}^d))^k$  acts as continuous linear functional on  $(C_c(\mathbb{R}^d))^k$  by

$$\langle \vec{f}, \vec{\nu} \rangle := \sum_{i=1}^k \langle f_i, \nu_i \rangle \quad \left( \vec{f} = (f_1, \dots, f_k) \in (C_c(\mathbb{R}^d))^k \right).$$

If  $H$  is any finite dimensional subspace of  $C_c(\mathbb{R}^d)$ , denote by  $\tilde{q}_H(\vec{\nu})$  the restriction of the linear functional  $\vec{\nu} \in (\mathcal{M}_p^+(\mathbb{R}^d))^k$  to the vector space  $H^k$ . Thus, formally,  $\tilde{q}_H(\vec{\nu})$  is an element of the dual  $H'_k$  of  $H^k$ . Now let  $Q_n$  denote the distribution of  $\varepsilon_n(X_{t_1}, \dots, X_{t_k})$  with respect to  $\mathbb{P}_{\mu/\varepsilon_n}^{\delta_n}$  ( $n \in \mathbb{N}$ ). Then, by (41) and Lemma 21, the conditions of the Gärtner-Ellis theorem (Theorem 2.3.6 in [DZ]) are fulfilled for the measures  $(Q_n \circ \tilde{q}_H^{-1})_{n \in \mathbb{N}}$ , provided  $H$  satisfies (40). Thus  $(Q_n \circ \tilde{q}_H^{-1})_{n \in \mathbb{N}}$  satisfies in  $H'_k$  a large deviation principle with speed  $(1/\varepsilon_n)_{n \in \mathbb{N}}$  and rate function  $\sup_{\vec{f} \in H^k} (\langle \vec{f}, \vec{\nu} \rangle - \log Z(\vec{f}))$  ( $\vec{\nu} \in H'_k$ ). Now denote by  $\mathbb{H}$  the class of all finite dimensional subspaces of  $C_c(\mathbb{R}^d)$  satisfying (40). If  $\tilde{H}, H \in \mathbb{H}$  satisfy  $\tilde{H} \subset H$ , let  $\tilde{p}_{\tilde{H}H} : H'_k \rightarrow \tilde{H}'_k$  denote the canonical restriction. Then  $(H'_k, \tilde{p}_{\tilde{H}H})_{\tilde{H} \subset H \in \mathbb{H}}$  is a projective system and, provided  $\mathbb{H}$  is rich enough, Lemma 19 tells us that  $(Q_n)_{n \in \mathbb{N}}$  satisfies a large deviation principle with rate function

$$\sup_{H \in \mathbb{H}} \sup_{\vec{f} \in H^k} \left( \langle \vec{f}, \vec{\nu} \rangle - \log Z(\vec{f}) \right) \quad \left( \vec{\nu} \in (\mathcal{M}_p^+(\mathbb{R}^d))^k \right).$$

Thus it remains for us to show that

$$(42) \quad \bigcup_{H \in \mathbb{H}} \left\{ \langle \vec{f}, \cdot \rangle \mid \vec{f} \in H^k \right\} \text{ separates the points of } (\mathcal{M}_p^+(\mathbb{R}^d))^k$$

and that, for  $\vec{\nu} \in (\mathcal{M}_p^+(\mathbb{R}^d))^k$

$$(43) \quad \sup_{\vec{f} \in (C_c(\mathbb{R}^d))^k} \left( \langle \vec{f}, \vec{\nu} \rangle - \log Z(\vec{f}) \right) = \sup_{H \in \mathbb{H}} \sup_{\vec{f} \in H^k} \left( \langle \vec{f}, \vec{\nu} \rangle - \log Z(\vec{f}) \right).$$

But let us first explain why even an arbitrary smooth function in  $C_c(\mathbb{R}^d)$  might not be contained in  $\bigcup_{H \in \mathbb{H}} H$ . To this end, it suffices to consider the case where  $k = 1$ . By Lemma 17, a function  $f \in C_c(\mathbb{R}^d)$  is in  $\partial \mathbf{D}_t$  if and only if  $\max_x f(x) = 1/t$ . The function  $V_t f$  then is singular on the set

of maximal points of  $f$ . However, it may occur that  $\log Z(f) = \langle V_t f, \mu \rangle < \infty$ . Only in the case where  $d = 1$  and  $\mu$  is Lebesgue measure, every continuously differentiable  $f \in \partial \mathbf{D}_t$  would satisfy  $Z(f) = \infty$ .

Our idea to overcome this difficulty is to use suitable polygonal approximations in the construction of the spaces  $H$ . Let us define, recursively in space dimension, a family  $A_\gamma^d$  ( $\gamma > 0$ ) of operators acting on  $C_c(\mathbb{R}^d)$ . If  $d = 1$ , we put for  $f \in C_c(\mathbb{R})$

$$A_\gamma^1 f(x) = \begin{cases} f(\gamma n) & \text{if } |\frac{x}{\gamma} - n| \leq \frac{1}{2} \\ & \text{for some } n \in 2\mathbb{Z}, \\ f(\gamma n)(n + \frac{3}{2} - \frac{x}{\gamma}) + f(\gamma(n+2))(\frac{x}{\gamma} - n - \frac{1}{2}) & \text{if } |\frac{x}{\gamma} - n - 1| \leq \frac{1}{2} \\ & \text{for some } n \in 2\mathbb{Z}. \end{cases}$$

For the definition of  $A_\gamma^{d+1}$  we rewrite  $x \in \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$  as  $x = (x_1, x_2)$  with  $x_1 \in \mathbb{R}^d$  and  $x_2 \in \mathbb{R}$ . Then let, for  $f \in C_c(\mathbb{R}^{d+1})$ ,  $A_\gamma^{d+1} f(x) = A_\gamma^1 \tilde{f}_{x_1}(x_2)$ , where, for  $z \in \mathbb{R}$ ,  $\tilde{f}_{x_1}(z) = (A_\gamma^d f(\cdot, z))(x_1)$ . Inductively one easily sees that  $A_\gamma^d$  is a linear operator mapping  $C_c(\mathbb{R}^d)$  into itself. Moreover, for every  $f \in C_c(\mathbb{R}^d)$ ,

$$(44) \quad \|A_\gamma f - f\|_\infty \longrightarrow 0 \quad \text{as } \delta \downarrow 0.$$

**Lemma 22** *Assume  $H_0$  is a finite dimensional subspace of  $C_c(\mathbb{R}^d)$  and  $\gamma > 0$ . Then the vector space  $H := \{A_\gamma^d f \mid f \in H_0\}$  satisfies (40).*

*Proof:* Let us introduce the notation  $g_{x,i}(t) = g(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d)$  ( $t \in \mathbb{R}$ ), where  $g$  is a function on  $\mathbb{R}^d$ ,  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$  and  $i = 1, \dots, d$ . We will say that a continuous function  $g : \mathbb{R}^d \rightarrow (-\infty, \infty]$  with compact support satisfies condition  $(C_\gamma)$ , if, for all  $x \in \mathbb{R}^d$  and  $i = 1, \dots, d$ , the function  $g_{x,i}$  is constant on intervals of the form  $[\gamma(n - \frac{1}{2}), \gamma(n + \frac{1}{2})]$  and convex on intervals of the form  $[\gamma(n - \frac{3}{2}), \gamma(n - \frac{1}{2})]$ , where  $n \in 2\mathbb{Z}$ . Obviously, every function of the form  $A_\gamma^d f$  satisfies  $(C_\gamma)$ .

If  $f_1, \dots, f_k$  satisfy  $(C_\gamma)$ , then so does  $f_1 + V_t f_2$  ( $t > 0$ ) and hence, inductively, the function  $v(x) := \log \mathbb{E}_{\delta_x}^0 \left[ \exp \left( \sum_{i=1}^k \langle f_i, X_{t_i} \rangle \right) \right]$ . Thus either  $v = \infty$  on some  $d$ -dimensional cube having edges of length  $\gamma$  and hence  $\langle v, \mu \rangle = \infty$ , or  $\max_{x \in \mathbb{R}^d} v(x) < \infty$ . But in the latter case we can find a constant  $\varepsilon > 0$ , such that  $\max_x \mathbb{E}_{\delta_x}^0 \left[ \exp \left( \sum_{i=1}^k \langle f_i + \varepsilon \phi_p, X_{t_i} \rangle \right) \right] < \infty$ . This implies  $Z(f_1 + \varepsilon \phi_p, \dots, f_k + \varepsilon \phi_p) < \infty$ , and hence the assertion is proved.  $\square$

Lemma 22 and (44) show that  $\bigcup_{H \in \mathbb{H}} H$  is dense in  $C_c(\mathbb{R}^d)$  with respect to the sup-norm. This immediately implies (42) and (43).

Now Theorem 2 is proved.  $\square$

### 9.3 Proof of Theorem 4:

**Lemma 23** For  $\nu, \eta \in \mathcal{M}_p^+(\mathbb{R}^d)$  denote by  $d\nu/d\eta$  the Radon-Nikodym derivative of  $\nu$  with respect to  $\eta$  in the extended sense of the Lebesgue decomposition. Let  $K(\nu | \eta)$  be given by

$$(45) \quad K(\nu | \eta) = \begin{cases} \int \left( \sqrt{\frac{d\nu}{d\eta}} - 1 \right)^2 d\eta + \nu \left( \frac{d\nu}{d\eta} = \infty \right) & \text{if } \text{supp } \nu \subset \text{supp } \eta, \\ \infty & \text{else.} \end{cases}$$

Then, for every  $\mu \in \mathcal{M}_p^+(\mathbb{R}^d)$  and each  $\omega \in C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$ ,

$$(46) \quad I_\mu^\sigma(\omega) = \sup_{0=t_0 < t_1 < \dots < t_n \leq 1} \sum_{i=1}^n \frac{1}{t_i - t_{i-1}} K(\omega(t_i) | \omega(t_{i-1})).$$

*Proof:* In view of Lemma 19, it suffices to show that

$$(47) \quad \sup_{f_0, \dots, f_n \in C_c(\mathbb{R}^d)} \left( \sum_{i=0}^n \langle f_i, \nu_i \rangle - \log \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=0}^n \langle f_i, X_{t_i} \rangle \right) \right] \right) = \sum_{i=1}^n \frac{1}{t_{i+1} - t_i} K(\nu_i | \nu_{i-1})$$

whenever  $0 = t_0 < t_1 < \dots < t_n$ ,  $\nu_1, \dots, \nu_n \in \mathcal{M}_p^+(\mathbb{R}^d)$  and  $\nu_0 = \mu$ .

This is trivial for  $n = 0$ . To prove the general case, we proceed by induction. The left hand side of (47) equals

$$\begin{aligned} & \sup_{f_0, \dots, f_n \in C_c(\mathbb{R}^d)} \left( \sum_{i=0}^n \langle f_i, \nu_i \rangle - \log \mathbb{E}_\mu^0 \left[ \exp \left( \sum_{i=0}^{n-1} \langle f_i, X_{t_i} \rangle + \langle V_{t_n - t_{n-1}} f, X_{t_{n-1}} \rangle \right) \right] \right) \\ &= \sum_{i=1}^{n-1} \frac{1}{t_{i+1} - t_i} K(\nu_i | \nu_{i-1}) + \sup_{f \in C_c(\mathbb{R}^d)} \left( \langle f, \nu_n \rangle - \langle V_{t_n - t_{n-1}} f, \nu_{n-1} \rangle \right). \end{aligned}$$

It is easy to calculate the Legendre transform of  $\alpha \mapsto V_{t_n - t_{n-1}} \alpha$ :

$$\sup_{\alpha \in \mathbb{R}} (\alpha \beta - V_{t_n - t_{n-1}} \alpha) = \frac{1}{t_n - t_{n-1}} (\sqrt{\beta} - 1)^2 \quad (\beta \geq 0).$$

A  $d$ -dimensional version of the main result in [LS] implies

$$(48) \quad \sup_{f \in C_c(\mathbb{R}^d)} \left( \langle f, \nu_n \rangle - \langle V_{t_n - t_{n-1}} f, \nu_{n-1} \rangle \right) = \frac{1}{t_n - t_{n-1}} K(\nu_n | \nu_{n-1}).$$

This proves the Lemma. A different proof of (48) has been given in Theorem 1.5.4 of [FK].  $\square$

It is not difficult to show that  $K(\nu | \eta)$  coincides with twice the square of the Kakutani-Hellinger distance. Hence the first part of Theorem 4 is proved. It remains to identify the Kakutani-Hellinger energy.

**Lemma 24** (i) If  $\Delta$  and  $\Delta'$  are dyadic partitions and  $\Delta \subset \Delta'$ , then  $\mathcal{E}_\Delta \leq \mathcal{E}_{\Delta'}$ .

(ii) If  $\mathcal{E}(\omega) < \infty$ , then  $t \mapsto \omega(t)$  is continuous with respect to the Kakutani-Hellinger distance  $d$  and  $\mathcal{E}(\omega) = \left\{ \mathcal{E}_\Delta(\omega) \mid \Delta \text{ is a dyadic partition} \right\}$ .

*Proof:* Observe that by Young's inequality  $d(\omega(t), \omega(r))^2 \leq 2(d(\omega(t), \omega(s))^2 + d(\omega(s), \omega(r))^2)$  for  $0 \leq r < s < t \leq 1$ . Setting  $s := (t - r)/2$  we thus get

$$\frac{d(\omega(t), \omega(r))^2}{t - r} \leq \frac{d(\omega(t), \omega(s))^2}{t - s} + \frac{d(\omega(s), \omega(r))^2}{s - r}.$$

This implies (i).

The first assertion of (ii) follows immediately from  $d(\omega(t), \omega(s))^2 \leq \mathcal{E}(\omega) \cdot |t - s|$ . Now choose a partition  $\Delta$  such that  $\mathcal{E}(\omega) \leq \mathcal{E}_\Delta(\omega) + \varepsilon/2$ . By continuity there exists a partition  $\Delta'$  consisting of dyadic rationals such that  $|\mathcal{E}_{\Delta'}(\omega) - \mathcal{E}_\Delta(\omega)| \leq \varepsilon/2$ . If  $\Delta''$  now is any dyadic partition containing  $\Delta'$ , it follows from the above that  $\mathcal{E}_{\Delta'}(\omega) \leq \mathcal{E}_{\Delta''}(\omega)$ . Hence  $\mathcal{E}(\omega) \leq \mathcal{E}_{\Delta''}(\omega) + \varepsilon$ , and the lemma is proved.  $\square$

Define for  $\omega \in C([0, 1] : \mathcal{M}_p^+(\mathbb{R}^d))$  and positive  $f \in C_c(\mathbb{R}^d)$  the  $C([0, 1] : M^+(\mathbb{R}^d))$ -valued path  $\omega_f$  by  $d\omega_f(t) := f d\omega(t)$  ( $0 \leq t \leq 1$ ). Now let  $(g_k)$  denote a partition of unity, and set  $f_n := \sum_{k=1}^n g_k$ . Then, for any  $\Delta$  and each  $n$ ,  $\mathcal{E}_\Delta(\omega_{f_n}) = \sum_{k=1}^n \mathcal{E}_\Delta(\omega_{g_k})$  and  $\mathcal{E}_\Delta(\omega) = \sum_{k=1}^\infty \mathcal{E}_\Delta(\omega_{g_k})$ . Hence Lemma 24 and monotone integration (applied to  $\sum_k$ ) imply that  $\mathcal{E}(\omega_{f_n}) = \sum_{k=1}^n \mathcal{E}(\omega_{g_k})$  and  $\mathcal{E}(\omega) = \sum_{k=1}^\infty \mathcal{E}(\omega_{g_k}) = \lim_n \mathcal{E}(\omega_{f_n})$ . Since also  $\int_0^1 \left\| \frac{d\dot{\omega}(t)}{d\omega(t)} \right\|_{L^2(\omega(t))} dt = \lim_n \int_0^1 \left\| \frac{d\dot{\omega}_{f_n}(t)}{d\omega_{f_n}(t)} \right\|_{L^2(\omega_{f_n}(t))} dt$ , we can restrict ourselves to the case where  $\omega \in C([0, 1] : M^+(\mathbb{R}^d))$  in order to prove Theorem 4.

First we will prove that  $\mathcal{E}(\omega)$  implies  $\omega \in \mathbf{H}$  and that  $\mathcal{E}(\omega) = \frac{1}{8} \int_0^1 \left\| \frac{d\dot{\omega}(t)}{d\omega(t)} \right\|_{L^2(\omega(t))} dt$ . To this end fix  $\omega \in C([0, 1] : M^+(\mathbb{R}^d))$  such that  $\mathcal{E}(\omega) < \infty$ .

**Lemma 25** (i) *The mapping  $t \mapsto \omega(t)$  is continuous in variation.*

(ii) *There exists  $\eta \in M^+(\mathbb{R}^d)$  such that  $\omega(t) \ll \eta$  for all  $t \in [0, 1]$ .*

*Proof:* By making use of  $\sup_t \|\omega(t)\|_{\text{var}} < \infty$ , (i) follows from Lemma 24 (ii) as in Proposition 4.4 in Chapter V of [JS].

To prove (ii), let  $\eta = \int_0^1 \omega(t) dt$ . Then  $\eta(A) = 0$  implies  $\omega(t)(A) = 0$  for almost every  $t$ . But  $t \mapsto \omega(t)(A)$  is continuous by (i), and hence  $\omega(t)(A) = 0$  for all  $t$ .  $\square$

Now fix  $\eta$  as above and let  $\varphi_t = \sqrt{d\omega(t)/d\eta}$ . With  $\|\cdot\|_p$  we will denote the norm in  $L^p(\eta) = L^p(\mathbb{R}^d, \eta)$ .

**Lemma 26**

$$\frac{1}{2} \int_0^1 \overline{\lim}_{h \rightarrow 0} \left\| \frac{\varphi_{t+h} - \varphi_t}{h} \right\|_2^2 dt \leq \mathcal{E}(\omega).$$

*Proof:* For  $0 \leq s < t \leq 1$  define  $J(s, t)$  by  $J(s, t) = \mathcal{E}(\omega(s \wedge \cdot \vee t))$ . Then  $t \mapsto J(0, t)$  is increasing and hence differentiable almost everywhere. Moreover, the derivative  $J'(t)$  satisfies  $0 \leq J'(t)$  for almost every  $t$  and  $\int_0^1 J'(t) dt \leq J(0, 1)$ . See e.g. [N], Chapter VIII, §2, Satz 4 and Satz 5. It follows easily from Lemma 24 that  $J(r, t) = J(r, s) + J(s, t)$  whenever  $0 \leq r < s < t \leq 1$ . Hence, for almost every  $t$ ,

$$\overline{\lim}_{h \rightarrow 0} \left\| \frac{\varphi_{t+h} - \varphi_t}{h} \right\|_2^2 \leq \lim_{h \rightarrow 0} \frac{1}{h} J(t, t+h) = J'(t),$$

and the Lemma is proved.  $\square$

**Lemma 27** *The mapping  $t \mapsto \varphi_t \in L^2(\eta)$  is weakly absolutely continuous and almost everywhere weakly differentiable with derivative  $\dot{\varphi}_t \in L^2(\eta)$ . In particular*

$$(49) \quad \frac{1}{2} \int_0^1 \|\dot{\varphi}_t\|_2^2 dt \leq \mathcal{E}(\omega).$$

*Proof:* Fix  $h \in L^2(\eta)$  with  $\|h\|_2 = 1$ . Then  $\int (\varphi_t - \varphi_s)^2 d\eta \geq \int h(\varphi_t - \varphi_s) d\eta$ , and thus

$$(50) \quad \infty > \mathcal{E}(\omega) \geq 2 \sup_{0 \leq t_0 < \dots < t_n \leq 1} \sum_{i=1}^n \left( \frac{\bar{x}(t_i) - \bar{x}(t_{i-1})}{t_i - t_{i-1}} \right)^2 (t_i - t_{i-1}),$$

where  $\bar{x}(t) = \int h\varphi_t d\eta$  ( $0 \leq t \leq 1$ ). But the right hand side of (50) would be infinite if  $\bar{x}$  were not absolutely continuous: see [DZ], p. 156.

Now Lemma 26 implies that, for almost every  $t$ ,  $(\varphi_{t+h} - \varphi_t)/h$  is weakly sequentially compact in  $L^2(\eta)$ . But there can be at most one limit point by the weak absolute continuity. This implies weak differentiability. Equation (49) finally follows from Lemma 26 and the weak lower semicontinuity of the norm.  $\square$

**Lemma 28** *The mapping  $t \mapsto \varphi_t \in L^2(\eta)$  is strongly differentiable almost everywhere,  $\dot{\varphi}$  is Bochner integrable, and it holds that*

$$(51) \quad \varphi_t = \varphi_0 + \int_0^t \dot{\varphi}_s ds \quad (0 \leq t \leq 1).$$

Moreover

$$(52) \quad \frac{1}{2} \int_0^1 \|\dot{\varphi}_t\|_2^2 dt = \mathcal{E}(\omega).$$

*Proof:* In view of Lemmata 26 and 27 the first part of the assertion and (51) are a consequence of Theorem 3.8.6 and Corollary 2 to Theorem 3.8.5 in [HP]. For (52) it remains to show the opposite inequality to (49). But

$$\begin{aligned} 2\mathcal{E}(\omega) &= \sup_{0 \leq t_0 < \dots < t_n \leq 1} \sum_{i=1}^n \left( \frac{\|\varphi_{t_i} - \varphi_{t_{i-1}}\|_2}{t_i - t_{i-1}} \right)^2 (t_i - t_{i-1}) \\ &\leq \sup_{0 \leq t_0 < \dots < t_n \leq 1} \sum_{i=1}^n \left( \frac{\int_{t_{i-1}}^{t_i} \|\dot{\varphi}_s\|_2 ds}{t_i - t_{i-1}} \right)^2 (t_i - t_{i-1}), \end{aligned}$$

which equals the left hand side of (52) (cf. [DZ], p. 156).  $\square$

**Lemma 29** *The mapping  $t \mapsto \varphi_t^2 \in L^1(\eta)$  is strongly differentiable almost everywhere, the derivative  $2\varphi_t \cdot \dot{\varphi}_t$  is Bochner integrable and it holds that  $\varphi_t^2 = \varphi_0^2 + 2 \int_0^t \varphi_s \dot{\varphi}_s ds$  ( $0 \leq t \leq 1$ ).*

*Proof:* By the argument in the first part of the proof of Lemma 28 we only have to show an analogue to Lemma 27 for  $\varphi_t^2$  in  $L^1(\eta)$ .

Suppose  $f \in L^\infty(\eta)$ . Then, with  $C := \max_{0 \leq t \leq 1} \|\varphi(t)\|_2$ ,

$$\left| \int f \varphi_t^2 d\eta - \int f \varphi_s^2 d\eta \right| \leq 2C \|f\|_\infty \cdot \|\varphi_t - \varphi_s\|_2 \leq 2C \|f\|_\infty \int_s^t \|\dot{\varphi}_u\|_2 du.$$

Therefore  $t \mapsto \varphi_t^2$  is indeed weakly absolutely continuous. It is now easy to see that  $\varphi_t^2$  possesses a weak derivative that satisfies

$$\int_0^1 \left\| \frac{d}{dt} \varphi_t^2 \right\|_1 dt = 2 \int_0^1 \|\varphi_t \dot{\varphi}_t\|_1 dt \leq 2 \left( \int_0^1 \|\varphi_t\|_2^2 dt \cdot \int_0^1 \|\dot{\varphi}_t\|_2^2 dt \right)^{1/2} < \infty.$$

□

Now, for almost every  $t$ , we can define a finite signed measure  $\dot{\omega}(t)$  by  $d\dot{\omega}(t) = 2\varphi_t \dot{\varphi}_t d\eta$ . Then  $\dot{\omega}(t) \ll \omega(t)$  by definition and  $\int \left( d\dot{\omega}(t)/d\omega(t) \right)^2 d\omega(t) = 4 \int \dot{\varphi}_t^2 d\eta$ . Lemma 29 assures that  $\omega(t) = \omega(0) + \int_0^t \dot{\omega}(s) ds$ . Thus  $\omega \in \mathbf{H}$  and

$$\mathcal{E}(\omega) = \frac{1}{8} \int_0^1 \left\| \frac{d\dot{\omega}(t)}{d\omega(t)} \right\|_{L^2(\omega(t))} dt.$$

It remains to show that  $\omega \in \mathbf{H}$  implies  $\mathcal{E}(\omega) < \infty$ .

**Lemma 30** *If  $\omega \in \mathbf{H} \cap C([0, 1] : M^+(\mathbb{R}^d))$  then  $t \mapsto \omega(t)$  is continuous in variation and  $\int_0^1 \|\dot{\omega}(t)\|_{\text{var}} dt < \infty$ .*

*Proof:* For  $0 \leq s < t \leq 1$

$$\begin{aligned} \|\omega(t) - \omega(s)\|_{\text{var}} &\leq \int_s^t \|\dot{\omega}(u)\|_{\text{var}} du = \int_s^t \left\| \frac{d\dot{\omega}(u)}{d\omega(u)} \right\|_{L^1(\omega(u))} du \\ &\leq \sup_{0 \leq r \leq 1} \sqrt{\|\omega(r)\|_{\text{var}}} \int_s^t \left\| \frac{d\dot{\omega}(u)}{d\omega(u)} \right\|_{L^2(\omega(u))} du. \end{aligned}$$

□

It follows easily from the next Lemma that  $\mathcal{E}(\omega) \leq \frac{1}{8} \int_0^1 \left\| d\dot{\omega}(t)/d\omega(t) \right\|_{L^2(\omega(t))}^2 dt < \infty$ , whenever  $\omega \in \mathbf{H}$ .

**Lemma 31** *Suppose  $\omega \in \mathbf{H}$ . Then  $d(\omega(t), \omega(s)) \leq \frac{1}{\sqrt{8}} \int_s^t \left\| d\dot{\omega}(t)/d\omega(t) \right\|_{L^2(\omega(t))} dt$ .*

*Proof:* Again we can restrict ourselves to the case where  $\omega \in C([0, 1] : M^+(\mathbb{R}^d))$ . Define  $\eta$  by  $\eta = \int_0^1 \omega(t) dt$  and conclude that  $\omega(t) \ll \eta$  for all  $t$  as in the proof of Lemma 25. Observe that  $\psi_t := d\dot{\omega}(t)/d\eta$  satisfies  $\|\psi_t\|_1 = \|\dot{\omega}(t)\|_{\text{var}}$  for almost every  $t$ . Thus it follows from Lemma 30 that  $t \mapsto \psi_t \in L^1(\eta)$  is Bochner integrable and that

$$\frac{d\omega(t)}{d\eta} - \frac{d\omega(s)}{d\eta} = \int_s^t \psi_u du \quad (0 \leq s < t \leq 1).$$

Now define functions  $f_n$  and  $f'_n$  on  $[0, \infty)$  by  $f'_n(x) = (1/2x^{1/2}) \wedge n$  and  $f_n(x) = \int_0^x f'_n(y) dy$  ( $n = 1, 2, \dots$ ). If  $\Delta = \{t_0, \dots, t_n\}$  is any dyadic partition of  $[s, t]$ , we conclude from the mean value theorem that

$$\begin{aligned} f_n\left(\frac{d\omega(t)}{d\eta}\right) - f_n\left(\frac{d\omega(s)}{d\eta}\right) &= \sum_{i=1}^n f'_n\left(\alpha_i \frac{d\omega(t_i)}{d\eta} + (1 - \alpha_i) \frac{d\omega(t_{i-1})}{d\eta}\right) \int_{t_{i-1}}^{t_i} \psi_u du \\ &=: \int_s^t g_n^\Delta(u) \psi_u du, \end{aligned}$$

where the  $\alpha_i$ 's are functions from  $\mathbb{R}^d$  to  $[0, 1]$ . It is easy to see that  $g_n^\Delta(u)$  converges boundedly to  $f'_n(d\omega(u)/d\eta)$  as  $\Delta$  becomes finer, and thus  $g_n^\Delta(u)\psi_u \rightarrow f'_n(d\omega(u)/d\eta)\psi_u$  in  $L^1(\eta)$  for almost every  $u$ . Furthermore  $\int_0^1 \|g_n^\Delta(t)\psi_t\|_1 dt \leq n \int_0^1 \|\psi_t\|_1 dt$  for all  $\Delta$ . From dominated convergence for Bochner integrals (Theorem 3.7.9 in [HP]) we therefore conclude that

$$(53) \quad f_n\left(\frac{d\omega(t)}{d\eta}\right) - f_n\left(\frac{d\omega(s)}{d\eta}\right) = \int_s^t f'_n\left(\frac{d\omega(u)}{d\eta}\right) \psi_u du.$$

But  $f_n(d\omega(u)/d\eta)$  also lies in  $L^2(\eta)$  and

$$(54) \quad \int_0^1 \left\| f'_n\left(\frac{d\omega(t)}{d\eta}\right) \psi_t \right\|_2 dt \leq \frac{1}{2} \int_0^1 \left\| \left(\frac{d\omega(t)}{d\eta}\right)^{-1/2} \psi_t \right\|_2 dt = \frac{1}{2} \int_0^1 \left\| \frac{d\dot{\omega}(t)}{d\omega(t)} \right\|_{L^2(\omega(t))} dt < \infty.$$

Therefore (53) also holds in  $L^2(\eta)$ . Finally, Fatou's Lemma and (54) imply that

$$\sqrt{2} \cdot d(\omega(t), \omega(s)) \leq \liminf_{n \uparrow \infty} \left\| f_n\left(\frac{d\omega(t)}{d\eta}\right) - f_n\left(\frac{d\omega(s)}{d\eta}\right) \right\|_2 \leq \frac{1}{2} \int_s^t \left\| \frac{d\dot{\omega}(u)}{d\omega(u)} \right\|_{L^2(\omega(u))} du.$$

□

**Acknowledgment:** The material in this article is based on a part of the author's doctoral thesis [S2]. I am grateful to my supervisor, H. Föllmer, for many helpful comments and constant encouragement. Further thanks are due to J. Brettschneider, K. Fleischmann and Th. Strobel for many useful remarks and stimulating discussions. Also I thank Yu. Kabanov for pointing out that the representation of  $I_\mu^0$  obtained in [S2] can be expressed in terms of the Kakutani-Hellinger distance.

## References

- [Az] Azencott, R., Grandes déviations et applications. In: Ecole d'Eté de Probabilités de Saint-Flour VIII, (Lect. Notes Math. vol. 774, pp. 1-176) Berlin etc.: Springer 1980

- [Br] Brettschneider, J.: Konstruktion der Austrittsmaße von Superprozessen und ihre Anwendung auf eine Klasse quasilinearer Dirichletprobleme. Diplomarbeit, Universität Bonn 1992
- [Da] Dawson, D. A.: Measure-Valued Markov Processes. In: Ecole d'Eté de Probabilités de Saint-Flour XXI, (Lect. Notes Math. vol. 1541, pp. 1-260) Berlin etc.: Springer 1993
- [DF] Dawson, D. A., Fleischmann, K.: Strong clumping of critical space-time branching models in subcritical dimensions. *Stoch. Proc. Appl.* **30**, 193-208 (1988)
- [DFG] Dawson, D. A., Fleischmann, K., Gorostiza, L.: Stable hydrodynamical limit fluctuations of a critical branching particle system in a random medium. *Ann. Prob.* **17**, 1083-1117 (1989)
- [DZ] Dembo, A., Zeitouni, O.: Large deviation techniques. Boston, London: Jones and Bartlett Publishers 1993
- [DW] Deuschel, J. D., Wang, K.: Large deviations for occupation time functionals of branching Brownian particles and super-Brownian motion. Preprint, ETH Zürich, 1993.
- [Dy] Dynkin, E. B.: Path processes and historical superprocesses. *Prob. Th. Related Fields* **90**, 1-36 (1991)
- [EKR] El Karoui, N., Roelly, S.: Propriétés de martingales, explosion et représentation de Lévy-Khinchine d'une classe de processus de branchement à valeurs mesures. *Stoch. Proc. Appl.* **38** 239-266 (1991)
- [FGK] Fleischmann, K., Gärtner, J., Kaj, I.: A Schilder type theorem for super-Brownian motion. Preprint, WIAS Berlin, 1994
- [FK] Fleischmann, K., Kaj, I.: Large deviation probabilities for some rescaled superprocesses. *Ann. Inst. Henri Poincaré* **30**, 607-645 (1994)
- [HP] Hille, E., Phillips, R.: Functional analysis and semi-groups. Providence: American Mathematical Society 1957
- [Is] Iscoe, I.: A weighted occupation time for a class of measure-valued critical branching processes. *Prob. Th. Rel. Fields* **71**, 85-116 (1986)
- [JS] Jacod, J., Shiryaev, A., N.: Limit theorems for stochastic processes. Berlin etc.: Springer 1987
- [KMM] Kerstan, J., Matthes, K., Mecke, J.: Unbegrenzt teilbare Punktprozesse. Berlin: Akademie-Verlag 1974
- [LG] Le Gall, J. F.: The Brownian snake and solutions of  $\Delta u = u^2$  in a domain. Preprint, Université Paris VI, 1994
- [LS] Lynch, J., Sethuraman, J.: Large deviations for processes with independent increments. *Ann. Prob.* **15**, 610-627 (1987)
- [N] Natanson, I., P.: Theorie der Funktionen einer reellen Veränderlichen. Thun: Verlag Harri Deutsch 1981
- [Pe] Perkins, E. A.: A space-time property of a class of measure-valued branching diffusions. *Trans. Amer. Math. Soc.* **305**, 743-795 (1988)
- [S1] Schied, A.: Criteria for exponential tightness in path spaces. Preprint, Universität Bonn, 1994

- [S2] Schied, A.: Große Abweichungen für die Pfade der Super-Brownschen Bewegung. Bonner Math. Schriften **277** (1995)
- [We] Weisler, F. B.: Existence and uniqueness of global solutions for a semilinear heat equation. Israel J. Math. **38**, 29-40 (1981)