Criteria for exponential tightness in path spaces

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Abstract: We derive criteria for exponential tightness in the space $C[0,1]$ and in the Skorohod space $D\big([0,1]:E\big)$, where E is a completely regular topological space. In the latter case we prove an "exponentially fast" version of Jakubowski's criterion, which enables one to reduce the problem to the exponential tightness of real valued stochastic processes. To deal with the space $C[0,1]$, we prove an embedding theorem between certain Hölder and Orlicz spaces in the spirit of Kolmogorov's criterion. As an application, we derive a Schilder-type result for the (measure-valued) Feller branching diffusion.

1. Introduction and statement of results

One of the crucial notions in the theory of large deviations is the concept of exponential tightness. It allows one, for example, to pass from a weak large deviation principle to a strong one or to refine the underlying topology of a given large deviation principle. The purpose of this paper is to give criteria for the exponential tightness of a sequence $X_\cdot^1, X_\cdot^2, \dots$ of stochastic processes, taking their values in some completely regular topological space E.

We shall assume that for each natural number $n \in \mathbb{N}$ the process $X_\cdot^n$ induces a measurable mapping from some given probability space $(\Omega, \mathcal{F}, P)$ into the Skorohod space

This work was supported in part by the Graduiertenkolleg "Algebraische, analytische und geometrische Methoden und ihre Wechselwirkung in der modernen Mathematik", Bonn.

AMS subject classification. 60F10

Key words and phrases. Exponential tightness, Hölder continuity, large deviations, Feller branching diffusion.

$D\big([0,1]:E\big)$, endowed with the Skorohod topology and its Borel field (cf.[6]). Note that this mapping is always measurable, if the process $X^n_\cdot$ is right continous with left limits and the space E is at least separable and metrizable.

Let (a_n) denote some sequence of positive real numbers, increasing to infinity. We will say that the sequence $(X^n_\cdot)$ is *exponentially tight* in $D\big([0,1]:E\big)$ with speed (a_n), if for each $L>0$ we can find a compact $K\subset D\big([0,1]:E\big)$ such that

$$\overline{\lim_{n\uparrow\infty}}\,\frac{1}{a_n}\log P\Big[X^n_\cdot\notin K\Big]\le -L.$$

To establish exponential tightness, we will adopt ideas which were developed to prove weak convergence of probability distributions on path spaces. In this case it turns out to be convenient to work with two different types of criteria. The first one reduces the problem to the tightness of a family of real valued processes plus some compact containment condition. The second criterion then applies to the one dimensional case. Here, usually, one has to estimate the modulus of continuity or the Hölder norm of the sample paths. Now we formulate an "exponentially fast" version of Jakubowski's tightness criterion:

Theorem 1

Let E be a completely regular topological space with metrizable compacts. Then the sequence $(X^n_\cdot)$ is exponentially tight in $D\big([0,1]:E\big)$ with speed (a_n), if and only if the following two conditions are fulfilled:

(i) For every $M>0$ there exists a compact $A_M\subset E$, such that

$$\overline{\lim_{n\uparrow\infty}}\,\frac{1}{a_n}\log P\Big[\exists\, t\;:\;X^n_t\notin A_M\Big]\le -M.$$

(ii) There exists an additive family $\mathbb{F}\subset C(E)$, separating the points of E, such that for each $f\in\mathbb{F}$ the sequence $\big(f(X^n_\cdot)\big)$ is exponentially tight in $D\big([0,1]:\mathbb{R}\big)$ with speed (a_n).

Remarks

1. The space $C\big([0,1]:E\big)$ of continous paths, endowed with the usual compact open topology, is a closed topological subspace of $D\big([0,1]:E\big)$ (cf.[6], Proposition 1.6 (i)). This shows that the above criterion holds in $C\big([0,1]:E\big)$ as well.
2. The assumption $\mathbb{F}\subset C(E)$ may be weakened. It suffices to assume that the restriction of each $f\in\mathbb{F}$ on each set A_M is continous.

3. The event $\Big\{\exists\, t\ :\ X_t \notin A\Big\}$ is Borel measurable if $A \subset E$ is closed (cf.[6], Proposition 1.6 (v)).

In our treatment of the one dimensional situation we will focus on the case of continuous processes.

For some $\alpha > 0$ and any $w \in C[0,1] := C\big([0,1] : \mathbb{R}\big)$ let

$$|w|_\alpha \ = \sup_{s \neq t} \frac{|w(t) - w(s)|}{|t-s|^\alpha}$$

denote the Hölder norm of w with exponent α. The Hölder space $H^\alpha[0,1]$ consists of all paths w with finite Hölder norm. More generally, for any Banach space E, we can define the space $H^\alpha\Big([0,1] : E\Big)$ of E-valued Hölder continuous paths.

In his well known criterion, Kolmogorov uses a moment condition to establish Hölder continuity of the sample paths of a stochastic process. Here we are going to give conditions, which guarantee that the Hölder norm of the sample paths even possesses exponential moments. To this end we will use the following generalization of the classical L^p-spaces: For $\kappa > 0$, let Φ_κ denote the function

$$\Phi_\kappa(x) = \frac{1}{\kappa}\Big(e^x - 1\Big) \qquad (x \geq 0).$$

The Luxemburg norm of a measurable function f on Ω with respect to Φ_κ then is given by

$$\|f\|_{\Phi_\kappa} \ = \inf\left\{\beta > 0 \,\middle|\, \int \Phi_\kappa\Big(\frac{|f|}{\beta}\Big)\, dP \leq 1\right\}.$$

Note that, if we replace Φ_κ by the mapping $x \mapsto x^p$, we would get the classical L^p-norm $\|f\|_p$. The space $L_{\Phi_\kappa}(\Omega, P)$ of all measurable functions on Ω with finite Luxemburg norm is usually called the Orlicz space with respect to Φ_κ. Here too, we can analogously define Orlicz spaces $L_{\Phi_\kappa}\Big(\Omega : E, P\Big)$ of mappings taking their values in some Banach space E. Our next result is the following embedding theorem which might be of independent interest:

Theorem 2

Let $\kappa \geq 1$, $0 < \alpha' < \alpha$ and let $|||\cdot|||_{\kappa,\alpha}$ denote the norm in $H^\alpha\Big([0,1] : L_{\Phi_\kappa}(\Omega, P)\Big)$. Then every process $X \in H^\alpha\Big([0,1] : L_{\Phi_\kappa}(\Omega, P)\Big)$ possesses a unique modification $\widetilde{X} \in L_{\Phi_\kappa}\Big(\Omega : H^{\alpha'}[0,1], P\Big)$, satisfying

$$\big\| \, |\widetilde{X}_\cdot|_{\alpha'} \big\|_{\Phi_\kappa} \leq c \cdot |||X|||_{\kappa,\alpha} \ ,$$

for some constant c, i.e., we have the continuous embedding

$$H^\alpha\Big([0,1] : L_{\Phi_\kappa}(\Omega, P)\Big) \hookrightarrow L_{\Phi_\kappa}\Big(\Omega : H^{\alpha'}[0,1], P\Big).$$

Moreover, the constant c may be choosen in such a way that it only depends on α and α':

$$c = \frac{(1 + p_0!) \cdot 2^{\alpha'+1}}{1 - 2^{-p_0^{-1}((\alpha-\alpha')p_0 - 1)}} \qquad \textit{with} \ \ p_0 := \left[\frac{1}{\alpha - \alpha'}\right] + 1$$

and with $[\,\cdot\,]$ denoting integer part.

Remarks:

1. Important ideas in the proof of Theorem 2 are taken from [3].
2. Theorem 2 extends to stochastic processes taking values in some Banach space or to continuous random fields with parameter space $[0,1]^d$. In the latter case one gets a slightly different embedding constant:

$$c = \frac{d \cdot (1 + p_0!) \cdot 2^{\alpha'+1}}{1 - 2^{-p_0^{-1}((\alpha-\alpha')p_0 - 1)}} \qquad \text{with} \ \ p_0 := \left[\frac{d}{\alpha - \alpha'}\right] + 1.$$

3. The classical Kolmogorov criterion can be stated in the same fashion as Theorem 2. In fact, inspecting the proof of Theorem 2.1 in [8], p. 25, one finds that

$$H^{\alpha+1/p}\Big([0,1] : L^p(\Omega, P)\Big) \hookrightarrow L^p\Big(\Omega : H^{\alpha'}[0,1], P\Big),$$

for all $p \geq 1$ and $0 < \alpha' < \alpha$.

As a corollary to Theorem 2 we get the following criterion for exponential tightness with respect to the compact-open topology in $C[0,1]$.

Theorem 3

The sequence $(X^n_\cdot)$ of real valued stochastic processes is exponentially tight in $C[0,1]$ with speed (a_n), if there are positive constants α, γ and κ such that, for all $s,t \in [0,1]$, $s \neq t$, we have

$$E\Big[\exp\Big(\frac{\gamma a_n}{|t-s|^\alpha}|X^n_t - X^n_s|\Big)\Big] \le \kappa^{a_n} \qquad \forall\, n \ge \text{ some } n_0,$$

and, additionally,

$$\lim_{R\uparrow\infty} \overline{\lim_{n\uparrow\infty}} \frac{1}{a_n} \log P\Big[\,|X^n_0| > R\Big] = -\infty.$$

In a forthcoming paper, [10], our results will be used to derive large deviation principles for the sample paths of rescaled super-Brownian motion. In order to illustrate the application of the Theorems 1 and 3, we will use them in the next two examples to give a simple proof of exponential tightness in Schilder's theorem and to state a large deviation principle for small perturbations of the (infinite dimensional) Feller branching diffusion.

Example 1 (Schilder's theorem)

Let (B_t) denote a Brownian motion with $P\big[B_0 = 0\big] = 1$. Schilder's theorem states that the sequence

$$B^n_t = \frac{1}{\sqrt{n}} B_t \qquad (0 \le t \le 1,\ n = 1, 2, \ldots)$$

of rescaled Brownian motions satisfies a large deviation principle in $C[0,1]$ (cf. [2]). We show exponential tightness with speed $a_n = n$ $(n = 1, 2, \ldots)$. For each $\beta > 0$ we have

$$\begin{aligned} E\Big[\exp\Big(\beta|B^n_t - B^n_s|\Big)\Big] &\le E\Big[\exp\Big(\frac{\beta}{\sqrt{n}}(B_t - B_s)\Big) + \exp\Big(\frac{\beta}{\sqrt{n}}(B_s - B_t)\Big)\Big] \\ &= 2 \cdot \exp\Big(\frac{\beta^2}{n}|t-s|\Big). \end{aligned}$$

Thus we get the following estimate

$$E\Big[\exp\Big(\frac{n}{\sqrt{|t-s|}}|B^n_t - B^n_s|\Big)\Big] \le 2e^n \le (2e)^n.$$

Choosing $\alpha = 1/2$, $\gamma = 1$ and $\kappa = 2e$, Theorem 3 implies exponential tightness. Analogously, one can prove exponential tightness in Schilder's theorem for the Brownian sheet.

Example 2 (Feller's diffusion)

The measure-valued Feller branching process is a diffusion taking values in $M(\mathbb{R}^d)$, the space of positive finite Borel measures on $\mathbb{R}^d$. It may be characterized by its Laplace functionals:

$$E\Big[\exp\big(\langle f, X_t\rangle\big)\,\Big|\,X_o=\mu\Big]=\exp\big(\langle V_t f,\mu\rangle\big), \tag{1}$$

where $f\in B_b(\mathbb{R}^d)$, the space of bounded and measurable functions on $\mathbb{R}^d$, and

$$V_t f(x)=\begin{cases}\dfrac{f(x)}{1-\rho t f(x)} & \text{if } f(x)<1/\rho t\\ \infty & \text{else}\end{cases} \tag{2}$$

for some fixed constant $\rho>0$. As usual, we set $\langle f,\nu\rangle := \int f\,d\nu$. Hence, this measure valued diffusion is the special case of a superprocess witout spatial motion, see [1].

Now let X_t^n $(0\le t\le 1)$ denote such a process with $X_0^n=\mu\in M(\mathbb{R}^d)$ a.s. and ρ replaced by ρ/n in (2) $(n=1,2,\ldots)$. In the sequel we will show that the sequence $(X_\cdot^n)$ satisfies a large deviation principle with speed $a_n=n$ in $C\big([0,1]:M(\mathbb{R}^d)\big)$. The corresponding rate function is given by

$$I(\omega)=\sup\bigg(\int_0^1\langle\phi(t,\cdot),\omega(t)\rangle\,dt-\log E\Big[\exp\Big(\int_0^1\langle\phi(t,\cdot),X_t\rangle\,dt\Big)\,\Big|\,X_0=\mu\Big]\bigg),$$

where $\omega\in C\big([0,1]:M(\mathbb{R}^d)\big)$ and the supremum is taken over all continuous functions ϕ on $[0,1]\times\mathbb{R}^d$ with compact support. However, it turns out that we can choose the weak topology for $M(\mathbb{R}^d)$ *only if μ has bounded support.* Otherwise we endow $M(\mathbb{R}^d)$ with the vague topology. In both cases, I is a good rate function.

This large deviation principle has independently been studied in [4], but the authors only used a weak topology on the underlying path space.

As a corollary, we find a large deviation principle for the total mass process $Y_t^n := \langle 1, X_t^n\rangle$ $(0\le t\le 1)$, wich is known to be a solution of the stochastic differential equation

$$dY_t^n=\sqrt{\frac{2\rho}{n}Y_t^n}\,dB_t \qquad Y_0^n=\langle 1,\mu\rangle \tag{3}$$

(cf. [1]). The corresponding rate function is given by

$$J(x)=\begin{cases}\dfrac{1}{\rho}\displaystyle\int_0^1\Big(\frac{d}{dt}\sqrt{x(t)}\Big)^2\,dt & \text{if } \sqrt{x(\cdot)}-1\in H,\\ \infty & \text{else,}\end{cases}$$

where H denotes the Cameron-Martin space (see [4] or [9]). Note that the diffusion coefficient $\sigma(x) = \sqrt{2\rho x}$ in (3) is too singular to get this result as a special case of the Freidlin-Wentzell theory.

Let us now turn to the proof of our assertions. As can be seen easily from (1) and (2), our large deviation principle may be taken as an infinite dimensional case of Cramér's theorem. Hence, by Theorem 6.1.3 in [2], at least a weak large deviation principle must hold. Thus, it suffices to show exponential tightness (cf. [2], Lemma 1.2.18). To this end, choose $g \in B_b(\mathbb{R}^d)$ and some $s, t \in [0,1]$, $s \neq t$. Using (1), (2) and the Markov property of the process (X_t), we observe that

$$\begin{aligned} &E\Big[\exp\Big(\big|\langle g, X_t\rangle - \langle g, X_s\rangle\big|\Big) \;\Big|\; X_0 = \mu\Big] \\ &\quad \le E\Big[\exp\Big(\langle g, X_t\rangle - \langle g, X_s\rangle\Big) + \exp\Big(\langle g, X_s\rangle - \langle g, X_t\rangle\Big) \;\Big|\; X_0 = \mu\Big] \\ &\quad \le 2\exp\Big\langle \frac{|t-s|\rho g^2}{1 - |t-s|\cdot\rho\cdot|g| - |t-s|\rho^2 g^2}, \mu\Big\rangle, \end{aligned}$$

provided g is small enough. Applying this estimate to the function

$$g(x) = \gamma n |t-s|^{-1/2} f(x),$$

where $f \in B_b(\mathbb{R}^d)$ is fixed and $\gamma > 0$ is a suitable constant, we finally get

$$E\Big[\exp\Big(\frac{\gamma n}{|t-s|^{1/2}}|\langle f, X_t^n\rangle - \langle f, X_s^n\rangle|\Big)\Big] \le \kappa^n \qquad \big(s, t \in [0,1],\, n = 1, 2, \ldots\big),$$

with

$$\kappa = 2\cdot\exp\Big\langle \frac{\rho(\gamma f)^2}{1 - \rho\gamma|f| - (\rho\gamma f)^2}, \mu\Big\rangle.$$

Therefore, by Theorem 2, the sequence $\langle f, X_\cdot^n\rangle$ $(n = 1, 2\ldots)$ is exponentially tight in $C[0,1]$ and condition (ii) of Theorem 1 is fulfilled, if we choose

$$\mathbb{F} = \Big\{\langle f, \cdot\rangle \;\Big|\; f \in C_b(\mathbb{R}^d)\Big\}.$$

In order to see that condition (i) is satisfied for the vague topology, apply the "only if" part of Theorem 1 with $E = [0, \infty)$ to the total mass processes $Y_\cdot^n = \langle 1, X_\cdot^n\rangle$ and recall that subsets of $M(\mathbb{R}^d)$ with bounded total mass are relatively compact w.r.t. the vague topology.

Now suppose the support A of μ is a bounded subset of $\mathbb{R}^d$. Then we claim that our process is supported by those measures wich do not charge A^c, the complement of A. To this end, observe that

$$E\Big[X_t(A^c)\,\Big|\,X_0=\mu\Big]=\mu(A^c)=0\qquad(t\geq 0)$$

and hence $X_t(A^c)=0$ a.s. for all $t\geq 0$. But, by Theorem 2, the process $\langle f,X_\cdot\rangle$ is Hölder continuous for each parameter $\alpha\in(0,1/2)$ and every $f\in B_b(\mathbb{R}^d)$ — as we have already shown. Hence $X_t(A^c)=0$ for all $t\geq 0$ a.s. Thus we can restrict our large deviation principle to the state space $M(A)$, where the vague and the weak topologies coincide.

Now suppose A is unbounded. Using ideas of [7], it is not difficult to show that the large deviations of the random measures X_1^n are governed by the rate function

$$J(\nu)=\begin{cases}\displaystyle\int\Big(\sqrt{\frac{d\nu}{d\mu}}-1\Big)^2d\mu+\nu\Big(\frac{d\nu}{d\mu}=\infty\Big) & \text{if } \operatorname{supp}\nu\subset A,\\ \infty & \text{else}\end{cases}\qquad\big(\nu\in M(\mathbb{R}^d)\big).$$

See [4] or [8] for details. But the rate function J obviously does not have compact level sets w.r.t. the weak topology. Using the contraction principle, we finally conclude that the sequence $X_\cdot^n$ $(n=1,2,\ldots)$ cannot be exponentially tight in $C\big([0,1]:M(\mathbb{R}^d)\big)$ when $M(\mathbb{R}^d)$ is furnished with the weak topology, and hence condition (i) of Theorem 1 must be violated.

2. Proofs

Proof of Theorem 1:

We prove Theorem 1 along the lines of [6].

The necessity of the two conditions can be shown as in the case of ordinary tightness.

To prove sufficiency, we note first that, by Lemma 3.2 in [6], the family $\mathbb{F}$ may be assumed to be countable:

$$\mathbb{F}\;=\;\{f_1,f_2,\ldots\}.$$

Now, for each $k\in\mathbb{N}$ and every $R>0$, there exists, by assumption, a compact $\widetilde{C}_R^k\subset D[0,1]$ with

$$\varlimsup_{n\uparrow\infty}\frac{1}{a_n}\log P\Big[f_k(X_\cdot^n)\notin\widetilde{C}_R^k\Big]\leq -R-1.$$

Hence there is some $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$P\Big[f_k(X^n_\cdot) \notin \widetilde{C}^k_R\Big] \leq e^{-Ra_n}.$$

Since $D[0,1]$ is polish, the set $\widetilde{C}^k_R$ may be enlarged to a compact C^k_R, satisfying

$$P\Big[f_k(X^n_\cdot) \notin C^k_R\Big] \leq e^{-Ra_n} \qquad \forall\, n \in \mathbb{N}.$$

Now let $L > 1$ be given. We define

$$K_L = \left\{ w \in D\big([0,1] : E\big) \,\middle|\, w(t) \in A_L \,\forall\, t,\ f_k\Big(w(\cdot)\Big) \in C^k_{kL} \,\forall\, k \in \mathbb{N} \right\}.$$

In [6] it is shown that K_L is a compact subset of $D\big([0,1] : E\big)$. So, to establish exponential tightness, it suffices to observe that

$$\begin{aligned}
\overline{\lim_{n\uparrow\infty}} \frac{1}{a_n} \log P\Big[X^n_\cdot \notin K_L\Big] &\leq (-L) \vee \left(\overline{\lim_{n\uparrow\infty}} \frac{1}{a_n} \log \sum_{k=1}^{\infty} P\Big[f^k(X^n_\cdot) \notin C^k_{kL}\Big]\right) \\
&\leq (-L) \vee \left(\overline{\lim_{n\uparrow\infty}} \frac{1}{a_n} \log \sum_{k=1}^{\infty} e^{-kLa_n}\right) \\
&\leq -L.
\end{aligned}$$

Proof of Theorem 2:
Let $H := |||X|||_{\kappa,\alpha}$. Then

$$\|X_t - X_s\|_{\Phi_\kappa} \leq H \cdot |t-s|^\alpha.$$

Since $\Phi_\kappa(x) \geq x^p/(\kappa p!)$ $(x \geq 0,\ p \in \mathbb{N})$, we get

$$\|\cdot\|_{\Phi_\kappa} \geq \frac{1}{\sqrt[p]{\kappa p!}} \|\cdot\|_p$$

with $\|\cdot\|_p$ denoting the usual L^p-norm. Hence

$$E\Big[|X_t - X_s|^p\Big] \leq p!\kappa H^p |t-s|^{p\alpha}.$$

Now we follow [3]. For $p \geq p_0$, a version of the classical Kolmogorov criterion implies the existence of a continous modification $\widetilde{X}$, satisfying

$$E\Big[|\widetilde{X}_\cdot|^p_{\alpha'}\Big] \leq p!\kappa(\widetilde{c}H)^p$$

with

$$\widetilde{c} = \frac{2^{\alpha'+1}}{1 - 2^{-p_0^{-1}((\alpha-\alpha')p_0-1)}}.$$

(cf. Theorem 2.1 in [8], p. 25 and its proof). In the case $p < p_0$ Jensen's inequality yields

$$E\Big[|\widetilde{X}_\cdot|_{\alpha'}^p\Big] \le E\Big[|\widetilde{X}_\cdot|_{\alpha'}^{p_0}\Big]^{\frac{p}{p_0}} \le \big(p_0!\kappa\big)^{\frac{p}{p_0}}(\widetilde{c}H)^p \le p_0!\,\kappa\widetilde{c}^p H^p.$$

By summing up, we get for any $\beta > 0$

$$\begin{aligned} E\Big[\Phi_\kappa\Big(\frac{|\widetilde{X}_\cdot|_{\alpha'}}{\beta}\Big)\Big] &\le p_0! \sum_{n=1}^{p_0-1} \frac{1}{n!}\Big(\frac{\widetilde{c}H}{\beta}\Big)^n + \sum_{n=p_0}^{\infty} \Big(\frac{\widetilde{c}H}{\beta}\Big)^n \\ &\le p_0! \frac{\widetilde{c}H}{\beta - \widetilde{c}H}. \end{aligned}$$

Now the right hand side is less than one, if

$$\beta \ge (p_0! + 1) \cdot \widetilde{c} \cdot H.$$

So, by definition of the Luxemburg norm, we conclude that

$$\big\||\widetilde{X}_\cdot|_{\alpha'}\big\|_{\Phi_\kappa} \le (p_0! + 1) \cdot \widetilde{c} \cdot H$$

and Theorem 2 is proved.

Proof of Theorem 3:

Our assumption implies

$$E\Big[\Phi_{\kappa^{a_n}}\Big(\frac{\gamma a_n}{|t-s|^\alpha}|X_t^n - X_s^n|\Big)\Big] \le 1$$

and hence

$$\big|\big|\big|X^n\big|\big|\big|_{\kappa^{a_n},\alpha} \le \frac{1}{\gamma a_n}.$$

Now, for any $\alpha' < \alpha$, we can apply Theorem 2 to get a constant c such that

$$\big\|\,|X_\cdot^n|_{\alpha'}\big\|_{\Phi_{\kappa^{a_n}}} \le \frac{c}{a_n}.$$

Note that c can be chosen independently of n. Consequently,

$$\begin{aligned}\frac{1}{a_n}\log P\Big[|X^n_\cdot|_{\alpha'} > L\Big] &\le \frac{1}{a_n}\log\left(e^{-\frac{La_n}{c}}E\left[\exp\left(\frac{a_n|X^n_\cdot|_{\alpha'}}{c}\right)\right]\right)\\ &\le \frac{1}{a_n}\log\left(e^{-\frac{La_n}{c}}(\kappa^{a_n}+1)\right)\\ &\le -\frac{L}{c}+\log(\kappa+1).\end{aligned}$$

This estimate yields the desired exponential tightness, since the set

$$\Big\{w \;\Big|\; |w|_{\alpha'} \le L,\, |w(0)| \le R\Big\}$$

is a compact subset of $C[0,1]$.

Acknowledgement: The material in this article is based on a part of the author's doctoral thesis [8]. I am grateful to my supervisor, Professor H. Föllmer, for many helpful comments and constant encouragement. Further thanks are due to Julia Brettschneider and Thomas Strobel for many useful remarks and stimulating discussions.

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