Optimal investments for robust utility functionals in complete market models

Alexander Schied*
Institut für Mathematik, MA 7-4
TU Berlin, Strasse des 17. Juni 136
10623 Berlin, Germany
e-mail: schied@math.tu-berlin.de

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Abstract: We introduce a systematic approach to the problem of maximizing the robust utility of the terminal wealth of an admissible strategy in a general complete market model, where the robust utility functional is defined by a set $\mathcal{Q}$ of probability measures. Our main result shows that this problem can often be reduced to determining a "least favorable" measure $Q_0 \in \mathcal{Q}$, which is universal in the sense that it does not depend on the particular utility function. The robust problem is thus equivalent to a standard utility maximization problem with respect to the "subjective" probability measure Q_0. By using the Huber-Strassen theorem from robust statistics, it is shown that Q_0 always exists if $\mathcal{Q}$ is the σ-core of a 2-alternating capacity. Besides other examples, we also discuss the problem of robust utility maximization with uncertain drift in a Black-Scholes market and the case of "weak information" as studied by Baudoin (2002).

1 Introduction

The problem of constructing utility-maximizing investment strategies in complete and incomplete market models has been a major theme of mathematical finance throughout the past decade. Today, the problem is very well understood, in particular through the efforts of Kramkov and Schachermayer [20], [21]; see also Karatzas and Shreve [19] and Schachermayer [25] for the history of the problem and an overview of further developments.

Economists, however, have long been arguing that the paradigm of von Neumann-Morgenstern expected utility, in both its objective and subjective forms, has various deficiencies. In its objective form, it requires precise knowledge of the probability distribution governing the market evolution, but this distribution is typically subject to *Knightian uncertainty*. In its subjective form, uncertainty is taken into account by means

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of a "subjective probability measure", but this approach is challenged by the celebrated Ellsberg paradox. In the late 1980's, Gilboa and Schmeidler [11], [28], [12] and Yaari [30] formulated natural axioms which should be satisfied by a preference order on payoff profiles in order to account for both risk and uncertainty aversion. They showed that such a preference order can be numerically represented by a *robust utility functional* of the form

$$X \longmapsto \inf_{Q \in \mathcal{Q}} E_Q[\, U(X)\,] , \tag{1}$$

where $\mathcal{Q}$ is a set of probability measures and U is a utility function.

In a financial market model, it is a natural objective for an investor to construct investment strategies that maximize the robust utility of the terminal wealth for a given initial amount of capital. In complete market models, first case studies of this problem were given by Baudoin [4] and Schied [26]. In this note, we now propose a systematic approach. More precisely, we give a complete solution to the problem of maximizing the robust utility of the terminal wealth in a complete market model, under the condition that the set $\mathcal{Q}$ admits a so-called "least favorable measure" Q_0. Our main result is that the robust problem is then equivalent to the standard utility maximization problem with respect to Q_0. Thus, although the preference order associated with (1) does not satisfy the axioms of (subjective) expected utility, optimal investment decisions are still made in accordance with the Savage/Anscombe-Aumann theory, provided that one takes Q_0 as "subjective" probability measure. Moreover, Q_0 is universal in the sense that is does not depend on the choice of any particular utility function. By means of the measure Q_0, we will also be able to translate the results by Kramkov and Schachermayer [20] and others to our robust setting.

We also discuss the existence and construction of the least favorable measure Q_0, which typically arises from $\mathcal{Q}$ in a non-linear way. For instance, if the set $\mathcal{Q}$ is the σ-core of a 2-alternating Choquet capacity, then Q_0 can be obtained by an application of the Neyman-Pearson lemma for capacities. This result was developed thirty years ago by Huber and Strassen [17] with the purpose of constructing optimal statistical tests for composite hypotheses and alternatives. The assumption that $\mathcal{Q}$ arises from a 2-alternating capacity is quite natural and includes examples such as convex distortions of probability measures or neighborhoods with respect to many standard probability metrics. We will also show that Baudoin's "weak information" [4] fits into this situation.

We also consider the problem of robust utility maximization in a standard Black-Scholes market with uncertain drift. Here, the set $\mathcal{Q}$ is not related to a 2-alternating capacity. Nevertheless, a least favorable measure Q_0 can be constructed by transforming the problem into a derivative pricing problem with *uncertain volatility* as discussed in El Karoui et al. [9]. We will also show that there may be no least favorable measure if we move beyond Black and Scholes towards stochastic volatility models.

This note is organized as follows. In the next section, we describe our model and the main results. Explicit examples are provided in Section 3: First we discuss robust utility maximization in a Black-Scholes market with uncertain drift. Then we recall the notion of a Radon-Nikodym derivative for capacities, and discuss several examples within the framework of the Huber-Strassen theory. In particular, we prove that the case of "weak information" corresponds to a 2-alternating capacity. Then we briefly review further examples from robust statistics. The proofs of our main results are given in Section 4.

2 Main results

We make the standard assumptions on our market model. That is, we consider a complete market model consisting of one bond and d risky assets, whose price processes are denoted by $S = (S_t^i)_{0\le t\le T, i=1,\dots,d}$. We may assume without loss of generality that the price of the bond is constant. The process S is assumed to be a semimartingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, and we emphasize that this includes the case of a discrete-time market model, in which prices are adjusted only at times $t = 0, 1, \dots, T$: just set $S_t := S_{[t]}$ and $\mathcal{F}_t := \mathcal{F}_{[t]}$ for arbitrary $t \in [0, T]$. We assume that $\mathcal{F}_0$ is P-trivial and that the market is complete in the sense that there exists a unique probability measure P^* that is equivalent to P and under which S is a d-dimensional local martingale. In a discrete-time setting, market completeness implies that Ω can be chosen as a finite set, and this will simplify certain assumptions on our set $\mathcal{Q}$.

A self-financing trading strategy can be regarded as a pair (x, ξ), where $x \in \mathbb{R}$ is the initial investment and $\xi = (\xi_t^i)_{0\le t\le T, i=1,\dots,d}$ is a predictable and S-integrable process. The value process X associated with (x, ξ) is given by $X_0 = x$ and

$$X_t = X_0 + \int_0^t \xi_r \, dS_r \,, \qquad 0 \le t \le T\,.$$

For $x \in \mathbb{R}$ given, we denote by $\mathcal{X}(x)$ the set of all such processes X with $X_0 \le x$ which are admissible in the sense that $X_t \ge 0$ for $0 \le t \le T$ and whose terminal wealth X_T has a well-defined robust utility

$$\inf_{Q\in\mathcal{Q}} E_Q[\, U(X_T)\,]$$

in the sense that

$$\inf_{Q\in\mathcal{Q}} E_Q[\, U(X_T) \wedge 0\,] > -\infty\,. \tag{2}$$

Here, $U : (0, \infty) \to \mathbb{R}$ is an increasing and strictly concave utility function. Now we can state our main problem:

$$\text{maximize } \inf_{Q\in\mathcal{Q}} E_Q[\, U(X_T)\,] \text{ among all } X \in \mathcal{X}(x). \tag{3}$$

Definition 2.1 *Let $\mathcal{Q}$ be a set of probability measures absolutely continuous with respect to P^*. $Q_0 \in \mathcal{Q}$ is called a* least favorable measure *with respect to P^* if the density $\pi = dP^*/dQ_0$ (taken in the sense of the Lebesgue decomposition) satisfies*

$$Q_0[\, \pi \le t\,] = \inf_{Q\in\mathcal{Q}} Q[\, \pi \le t\,] \qquad \textit{for all } t > 0.$$

In the sequel, we will assume that $\mathcal{Q}$ is a convex set. Moreover, we will assume throughout this note that $\mathcal{Q}$ is equivalent to P^* in the following sense:

$$P^*[\,A\,] = 0 \qquad \Longleftrightarrow \qquad Q[\,A\,] = 0 \text{ for all } Q \in \mathcal{Q}. \tag{4}$$

Clearly, our problem (3) would not be well-posed without the implication "$\Rightarrow$". The converse implication is economically natural, since a position with a positive price should lead to a non-vanishing utility. Note that (4) is strictly weaker than the condition that *every* measure in $\mathcal{Q}$ is equivalent to P^*, which is often assumed in papers on model uncertainty; for a discussion see Cont [7].

Now we can state our first main result. It reduces the robust utility maximization problem to a standard utility maximization problem plus the computation of a least favorable measure, which is *independent* of the utility function.

Theorem 2.2 *Suppose that $\mathcal{Q}$ admits a least favorable measure $Q_0 \approx P^*$. Then the robust utility maximization problem* (3) *is equivalent to the standard utility maximization problem with respect to Q_0, i.e., to* (3) *with $\mathcal{Q}$ replaced by $\mathcal{Q}_0 := \{Q_0\}$. More precisely, $X_T^* \in \mathcal{X}(x)$ solves the robust problem* (3) *if and only if it solves the standard problem for Q_0, and the corresponding value functions are equal, whether there exists a solution or not:*

$$\sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[\,U(X_T)\,] = \sup_{X \in \mathcal{X}(x)} E_{Q_0}[\,U(X_T)\,], \quad \textit{for all } x.$$

This result has the following striking economic consequence. Let $\succ$ denote the preference order induced by our robust utility functional, i.e.,

$$X \succ Y \qquad \Longleftrightarrow \qquad \inf_{Q \in \mathcal{Q}} E_Q[\,U(X)\,] > \inf_{Q \in \mathcal{Q}} E_Q[\,U(Y)\,]\,.$$

Then, although $\succ$ does not satisfy the axioms of (subjective) expected utility theory, optimal investment decisions with respect to $\succ$ are still made in accordance with the Savage/Anscombe-Aumann version of expected utility, provided that we take Q_0 as the subjective probability measure.

By combining Theorem 2.2 with Proposition 3.1 below, we are able to translate [20, Theorem 2.0] to our situation. To this end, we have to assume that U is continuously differentiable and satisfies the Inada conditions

$$U'(0) := \lim_{x \downarrow 0} U'(x) = +\infty \qquad \text{and} \qquad U'(\infty) := \lim_{x \uparrow \infty} U'(x) = 0\,.$$

We denote by

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[\,U(X_T)\,]\,, \qquad x > 0\,,$$

the value function of the problem (3). Since $u(x) \geq U(x)$ for all x, our condition (2) on $\mathcal{X}(x)$ poses no restriction. Let

$$V(y) = \sup_{x > 0} \left[U(x) - xy \right], \qquad y > 0\,,$$

denote the convex conjugate of U and define the function

$$I := -V' = (U')^{-1}.$$

We also define the convex function

$$v(y) = \inf_{Q \in \mathcal{Q}} E_Q[\, V(y \cdot \pi)\,]\,, \qquad y > 0\,.$$

Corollary 2.3 *Suppose that $\mathcal{Q}$ admits a least favorable measure $Q_0 \approx P^*$ and that $u(x)$ is finite for some $x > 0$. Then:*

(a) *$u(x)$ is finite for all $x > 0$, and $v(y) < \infty$ for $y > 0$ sufficiently large. The function v is continuously differentiable in the interior (y_0, ∞) of its effective domain. The function u is continuously differentiable on $(0, \infty)$ and strictly concave on $(0, x_0)$, where $x_0 := -\lim_{y \downarrow y-0} v'(y)$. For $x, y > 0$,*

$$v(y) = \sup_{x>0} \big[u(x) - xy\big] \qquad \text{and} \qquad u(x) = \inf_{y>0} \big[v(y) + xy\big]\,.$$

Moreover, $u'(0) := \lim_{x \downarrow 0} u'(x) = \infty$ and $v'(\infty) = \lim_{y \uparrow \infty} v'(y) = 0$.

(b) *For $x < x_0$ there exists a unique solution $X^*(x) \in \mathcal{X}(x)$ of (3), and its terminal wealth is of the form*

$$X^*_T(x) = I(y \cdot \pi)\,, \qquad \text{for } y = u'(x).$$

(c) *For $0 < x < x_0$ and $y < y_0$,*

$$u' = x^{-1} \sup_{Q \in \mathcal{Q}} E_Q\big[\, X^*_T(x) U'(X^*_T(x))\,\big] \qquad \text{and} \qquad v'(y) = E^*[\, V'(y \cdot \pi)\,]\,.$$

Kramkov and Schachermayer [20], [21] give further results on optimal investment strategies, in particular those involving the asymptotic elasticity of U and necessary conditions for the validity of the duality theorem. We leave it to the reader to translate the complete-market versions of these theorems to our robust setting.

Motivated by an earlier version of this paper, Gundel [13] has studied the case in which $\mathcal{Q}$ does not necessarily admit a least favorable measure. She gives conditions under which a similar result as Corollary 2.3 holds for a measure $Q_0 \in \mathcal{Q}$, which then will depend on both the utility function U and the initial investment x. She also obtains duality results in incomplete market models. However, in [13] our condition (2) is replaced by the stronger requirement that *every* measure $Q \in \mathcal{Q}$ is equivalent to a given reference measure P, which rules out many of the examples in our Section 3 below. An extension of Corollary 2.3 to incomplete markets without the restrictions in [13] has recently been obtained in Schied and Wu [27].

Let us now show that the condition $Q_0 \approx P^*$ is always satisfied if $\mathcal{Q}$ is convex and closed in total variation. Recall that $\mathcal{Q}$ is closed in total variation if and only if $\{\, dQ/dP^* \,|\, Q \in \mathcal{Q}\,\}$ is closed in $L^1(P^*)$.

Lemma 2.4 *Suppose that $\mathcal{Q}$ is convex and and closed in total variation. Then every least favorable measure Q_0 is equivalent to P^*.*

Proof: Due to our assumptions and the Halmos-Savage theorem, $\mathcal{Q}$ contains a measure $Q_1 \approx P^*$. We get

$$1 = Q_0[\,\pi < \infty\,] = \lim_{t\uparrow\infty} Q_0[\,\pi \le t\,] = \lim_{t\uparrow\infty} \inf_{Q\in\mathcal{Q}} Q[\,\pi \le t\,] \le Q_1[\,\pi < \infty\,].$$

Hence, also $P^*[\,\pi < \infty\,] = 1$ and in turn $P^* \ll Q_0$. □

Let us conclude this section by stating the following converse to Theorem 2.2, which was suggested by an anonymous referee:

Theorem 2.5 *Suppose $Q_0 \in \mathcal{Q}$ is such that for all utility functions and all $x > 0$ the robust utility maximization problem* (3) *is equivalent to the standard utility maximization problem with respect to Q_0. Then Q_0 is a least favorable measure in the sense of Definition 2.1.*

The proof will show that in the preceding Theorem the class of all utility functions can be replaced by the smaller class of all bounded and continuously differentiable utility functions. The proofs of Theorems 2.2 and 2.5 will be given in Section 4.

3 Examples

In this section, we will discuss three classes of examples in which least favorable measures can be determined. The first is a Black-Scholes market with uncertain drift. The second is provided by the classical Huber-Strassen theory, where $\mathcal{Q}$ is is the σ-core of a 2-alternating capacity. The third class is given by extensions of the Huber-Strassen theory due to Huber [16] and Augustin [1].

First, let us state the following elementary characterization of least favorable measures, which is a variant of [17, Theorem 6.1].

Proposition 3.1 *For $Q_0 \in \mathcal{Q}$ with $Q_0 \approx P^*$ and $\pi := dP^*/dQ_0$, the following conditions are equivalent.*

(a) Q_0 is a least favorable measure for P^*.

(b) For all decreasing functions $f : (0,\infty] \to \mathbb{R}$ such that $\inf_{Q\in\mathcal{Q}} E_Q[\,f(\pi) \wedge 0\,] > -\infty$,

$$\inf_{Q\in\mathcal{Q}} E_Q[\,f(\pi)\,] = E_{Q_0}[\,f(\pi)\,]\,.$$

(c) For all increasing functions $g : (0,\infty] \to \mathbb{R}$ such that $\sup_{Q\in\mathcal{Q}} E_Q[\,g(\pi) \vee 0\,] < \infty$,

$$\sup_{Q\in\mathcal{Q}} E_Q[\,g(\pi)\,] = E_{Q_0}[\,g(\pi)\,]\,.$$

(d) Q_0 minimizes

$$I_\Phi(P^*|Q) := \int \Phi\Big(\frac{dQ}{dP^*}\Big)\,dP^*$$

among all $Q \in \mathcal{Q}$, for all continuous convex functions $\Phi : [0,\infty) \to \mathbb{R}$ such that $I_\Phi(P^*|Q)$ is finite for some $Q \in \mathcal{Q}$.

Proof: (a)$\Leftrightarrow$(b): According to the definition, Q_0 is a least favorable measure if and only if $Q_0 \circ \pi^{-1}$ stochastically dominates $Q \circ \pi^{-1}$ for all $Q \in \mathcal{Q}$. Hence, if f is bounded, then the equivalence of (a) and (b) is just the standard characterization of stochastic dominance (see, e.g., [10, Theorem 2.71]). If f is unbounded but satisfies $\inf_{Q \in \mathcal{Q}} E_Q[\, f(\pi) \wedge 0\,] > -\infty$, then assertion (b) holds for $f_N := (-N) \vee f \wedge 0$. Thus, for all $Q \in \mathcal{Q}$ and $N \in \mathbb{N}$,

$$E_Q[\, f_N(\pi)\,] \geq E_{Q_0}[\, f_N(\pi)\,] \geq E_{Q_0}[\, f(\pi) \wedge 0\,] > -\infty\,.$$

By sending N to infinity, it follows that $E_Q[\, f(\pi) \wedge 0\,] \geq E_{Q_0}[\, f(\pi) \wedge 0\,]$ for every $Q \in \mathcal{Q}$. After using a similar argument on $0 \vee f(\pi)$, we get

$$E_Q[\, f(\pi)\,] = E_Q[\, f(\pi) \vee 0\,] + E_Q[\, f(\pi) \wedge 0\,] \geq E_{Q_0}[\, f(\pi)\,] \qquad \text{for all } Q \in \mathcal{Q}.$$

(b)$\Leftrightarrow$(c) follows by changing signs.

(b)$\Rightarrow$(d): Clearly, $I_\Phi(P^*|Q)$ is well-defined and larger than $\Phi(1)$ for each $Q \ll P$. Now take a $Q_1 \in \mathcal{Q}$ with $I_\Phi(P^*|Q_1) < \infty$, and denote by $\Phi'_+(x)$ the right-hand derivative of Φ at $x \geq 0$. Suppose first that Φ'_+ is bounded. Since $\Phi(y) - \Phi(x) \geq \Phi'_+(x)(y-x)$, we have

$$I_\Phi(P^*|Q_1) - I_\Phi(P^*|Q_0) \geq \int \Phi'_+\big(\pi^{-1}\big)\Big(\frac{dQ_1}{dP^*} - \frac{dQ_0}{dP^*}\Big)\, dP^* = \int f(\pi)\, dQ_1 - \int f(\pi)\, dQ_0\,,$$

where $f(x) := \Phi'_+(1/x)$ is a bounded decreasing function. Therefore $\int f(\pi)\, dQ_1 \geq \int f(\pi)\, dQ_0$, and Q_0 minimizes $I_\Phi(P^*|\,\cdot\,)$ on $\mathcal{Q}$. If Φ'_+ is unbounded, one can either use a straightforward approximation argument or apply [10, Corollary 2.62].

(d)$\Rightarrow$(b): It is enough to prove (b) for continuous bounded decreasing functions f. For such a function f let $\Phi(x) := \int_1^x f(1/t)\, dt$. Then Φ is convex. For $Q_1 \in \mathcal{Q}$ we let $Q_t := tQ_1 + (1-t)Q_0$ and $h(t) := I_\Phi(P^*|Q_t)$. The right-hand derivative of h satisfies $0 \leq h'_+(0) = \int f(\pi)\, dQ_1 - \int f(\pi)\, dQ_0$, and the proof is complete. $\square$

Remark 3.2 By taking a strictly convex function Φ in (d), it follows that there exists at most one equivalent least favorable measure Q_0. If condition (4) is dropped, then there may be several least favorable measures; see the proof of Proposition 3.13 for examples.

3.1 Utility maximization with uncertain drift

Consider a Black-Scholes market model with a riskless bond, B_t, of which we assume $B_t \equiv 1$ and with d risky assets $S_t = (S_t^1, \dots, S_t^d)$ that satisfy an SDE of the form

$$dS_t^i = S_t^i \sum_{j=1}^d \sigma_t^{ij}\, dW_t^j + \alpha_t^i S_t^i\, dt \tag{5}$$

with a d-dimensional Brownian motion W and a volatility matrix σ_t that has full rank. Now suppose the investor is uncertain about the “true” future drift $\alpha_t = (\alpha_t^1, \dots, \alpha_t^d)$ in the market: any drift α is possible that is adapted to the filtration generated by W and satisfies $\alpha_t \in C_t$, where C_t is a nonrandom bounded closed convex subset of $\mathbb{R}^d$. Let us denote by $\mathcal{A}$ the set of all such processes α. This uncertainty in the choice of the drift can be expressed by the set

$$\mathcal{Q} := \Big\{ Q \,\Big|\, S \text{ has drift } \alpha^Q \in \mathcal{A} \text{ under } Q \Big\}.$$

Under P^* the drift α in (5) vanishes. It turns out that the optimal investment problem with *uncertain drift* can be solved by transforming it into a problem for *uncertain volatility* as studied by El Karoui et al. [9]. To this end, we denote by α^0_t the element in C_t that minimizes the norm $|\sigma_t^{-1}x|$ among all $x \in C_t$

Proposition 3.3 *Suppose that σ_t is deterministic and that both α^0_t and σ_t are continuous in t. Then $\mathcal{Q}$ admits a least favorable measure Q_0 with respect to P^*, which is characterized by having the drift α^0.*

Proof: We will use arguments from [9] to check condition (d) of Proposition 3.1. The density process of $Q \in \mathcal{Q}$ with respect to P^* has the form

$$Z^Q_t := \frac{dQ}{dP^*}\Big|_{\mathcal{F}_t} = \exp\Big(\int_0^t \lambda_s \, dW^*{}_s - \frac{1}{2}\int_0^t |\lambda_s|^2 \, ds \Big),$$

where $\lambda_s = \sigma_s^{-1}\alpha^Q_s$ and W^* is a d-dimensional P^*-Brownian motion. Similarly, the density process $Z := Z^{Q_0}$ will involve the deterministic integrand $\gamma_s := \sigma_s^{-1}\alpha^0_s$. Let Φ be a convex function on $\mathbb{R}_+$. We may assume without loss of generality that Φ has at most polynomial growth. Then $v(t,x) := E^*[\,\Phi(xZ_t)\,]$ is a solution of the Black-Scholes equation $v_t = \frac{1}{2}|\gamma_t|^2 x^2 v_{xx}$. This fact and Itô's formula show that

$$dv(T-t, Z^Q_t) = v_x(T-t, Z^Q_t)\, dZ^Q_t + \frac{1}{2}(Z^Q_t)^2 v_{xx}(T-t, Z^Q_t)(|\lambda_t|^2 - |\gamma_t|^2)\, dt\,.$$

One easily checks that the first term on the right is a martingale increment. Moreover, v is convex and $|\lambda_t|^2 \geq |\gamma_t|^2$ by definition of α^0. Hence, $v(T-t, Z^Q_t)$ is a submartingale and

$$E^*[\,\Phi(Z^Q_T)\,] = E^*[\,v(0, Z^Q_T)\,] \geq v(T, Z^Q_0) = E^*[\,\Phi(Z_T)\,]\,.$$

□

An obvious question is whether the strong condition that the volatility σ_t and the drift α^0 are deterministic can be relaxed. One case of interest would be a local volatility model in which the equation (5) is replaced by the one-dimensional SDE

$$dS_t = \sigma(t, S_t)S_t \, dW_t + \alpha_t S_t \, dt\,. \tag{6}$$

In this case, however, the density process Z appearing in the preceding proof involves the integrand $\gamma_t = \sigma(t,S_t)^{-1}\alpha^0_t$, which depends in a nontrivial way on the whole path of W. By using arguments that are due to M. Yor and reported in Section 4 of [9], we will show below that Proposition 3.3 may break down if γ is path-dependent, and this may occur if either σ or α^0 are not deterministic. Moreover, $\sigma(t, S_t)$ is not Hölder continuous of order

1/2, and so the method developed by Hajek [14], Hobson [15], and Janson and Tysk [18] does not apply. It is therefore not clear to the writer whether Proposition 3.3 remains true for the SDE (6).

For simplicity, let us continue the discussion in dimension $d = 1$. Then

$$S_t = S_0 \exp\Big(\int_0^t \sigma_s\,dW_s + \int_0^t \Big(\alpha_s - \frac{1}{2}\sigma_s^2\Big)\,ds\Big) \tag{7}$$

solves $dS_t = \sigma_t S_t\,dW_t + \alpha_t S_t\,dt$ whenever σ and α are appropriately integrable, and the model will be complete as soon as α and σ are adapted to the filtration generated by W and α_t/σ_t is bounded. Furthermore, each set C_t is equal to a closed interval $[c_t^0, c_t^1]$. It is economically reasonable to assume $c_t^0 > 0$ (for $\alpha_t < 0$ the risky asset would not be traded by any risk-averse investor), and in this case α_t^0 is simply given by c_t^0. The measures $Q \in \mathcal{Q}$ will then correspond to the laws of the processes S defined by (7), where α is adapted and satisfies a.s. $c_t^0 \le \alpha_t \le c_t^1$. The measure P^* corresponds to $\alpha \equiv 0$.

In the following proposition, we will give examples in which this set $\mathcal{Q}$ does *not* admit least favorable measures. To this end, let us fix some level $a > 0$, and denote by $T_a := \inf\{\, t \ge 0 \,|\, W_t = a\,\}$ the first time at which W hits the level a.

Proposition 3.4 *There exist $\varepsilon > 0$ and $0 < x < a$ such that:*

(a) *For $c_t^0 \equiv 1$, $c_t^1 \equiv 1 + \varepsilon^{-1}$, and*

$$\sigma_t := \frac{1}{\varepsilon + \mathrm{I}_{\{W_t < x,\, t \le T_a\}}},$$

the set $\mathcal{Q}$ does not admit a least favorable measure with respect to P^*.

(b) *For $\sigma_t \equiv 1$, $c_t^0 = \varepsilon + \mathrm{I}_{\{W_t < x,\, t \le T_a\}}$, and $c_t^1 \equiv 1 + \varepsilon$, the set $\mathcal{Q}$ does not admit a least favorable measure with respect to P^*.*

(c) *For $\sigma_t \equiv 1$, $c_t^0 \equiv \varepsilon$, and $c_t^1 \equiv 1 + \varepsilon$, $\mathcal{Q}$ does not admit a least favorable measure with respect to the law $\widetilde{P}$ of (7) with $\alpha_t = \mathrm{I}_{\{W_t < x,\, t \le T_a\}}$.*

Proof: In the situation of part (a), we can define measures $Q_y \in \mathcal{Q}$ by taking the drift

$$\alpha_t^y := \sigma_t\big(\varepsilon + \mathrm{I}_{\{W_t < x+y,\, t \le T_a\}}\big) \le 1 + \varepsilon^{-1},$$

for $y \ge 0$. The choice $y = 0$ corresponds to the minimal drift α^0. Consider the convex function $\Phi(z) = (z - ae^a)^+$. A straightforward modification of the arguments in Section 4 of [9] shows that $x \in (0, a)$ and $\varepsilon > 0$ may be chosen in such a way that

$$I_\Phi(P^*|Q_0) = E^*\big[\,(Z_T^{Q_0} - ae^a)^+\big] > E^*\big[\,(Z_T^{Q_{a-x}} - ae^a)^+\big] = I_\Phi(P^*|Q_{a-x}),$$

where

$$Z_T^{Q_y} = \exp\Big(\int_0^T \big(\varepsilon + \mathrm{I}_{\{W_t < x+y,\, t \le T_a\}}\big)\,dW_t - \frac{1}{2}\int_0^T \big(\varepsilon + \mathrm{I}_{\{W_t < x+y,\, t \le T_a\}}\big)^2\,dt\Big).$$

Hence, due to Proposition 3.1, Q_0 cannot be a least favorable measure. However, taking the convex function $\Psi(z) = -\log z$ yields

$$I_\Psi(P^*|Q) = \frac{1}{2}E^*\Big[\int_0^T \Big(\frac{\alpha_t}{\sigma_t}\Big)^2\,dt\Big], \qquad Q \in \mathcal{Q},$$

and this expression is minimized by Q_0. Thus, there can be no least favorable measure, and (a) follows. The proofs for parts (b) and (c) are similar. □

Remark 3.5 When $d = 1$ and $\mathcal{A}$ is of the form $\mathcal{A} = \{\,\widetilde{\alpha}\,\big|\,|\lambda_t - \widetilde{\alpha}_t/\sigma_t| \le \beta_t \text{ a.e.}\,\}$, the upper and lower expectations induced by the corresponding set $\mathcal{Q}$ can be interpreted as *g-expectations* in the sense of Peng [24]; see, e.g., Example 1 of Chen and Sulem [5].

3.2 Examples within the Huber-Strassen theory

In the preceding section, the way of determining the set $\mathcal{Q}$ was to specify a "confidence set" around an estimate of a certain market parameter and to take for $\mathcal{Q}$ the class of all measures that are consistent with this confidence set. In practice, however, one would rather try to assign a high weight to the original estimate, while a measure concentrated on the outmost edge of the confidence set should receive a lower weight. This idea illustrates that the set $\mathcal{Q}$ may arise in a more complicated manner from the investor's preference relation than in the ad hoc approach of the preceding section.

The complexity of determing the set $\mathcal{Q}$ is reduced if one imposes additional assumptions on the underlying preference order. For instance, Schmeidler [28] introduced the assumption of *comonotonic indepence*, which is reasonable insofar comonotonic positions cannot act as mutual hedges; see [28], p. 576, for a more detailed economic justification of comonotonic independence. Mathematically, comonotonic independece is essentially equivalent to the fact that the nonadditive set function

$$\gamma(A) := \sup_{Q\in\mathcal{Q}} Q[\,A\,], \qquad A \in \mathcal{F}_T,$$

is 2-alternating in the sense of Choquet:

$$\gamma(A\cup B) + \gamma(A\cap B) \le \gamma(A) + \gamma(B) \qquad \text{for } A, B \in \mathcal{F}_T;$$

see [28], p. 582.

Assumption 3.6 *Consider the following set of conditions.*

(a) *γ is 2-alternating.*

(b) *$\mathcal{Q}$ is maximal in the sense that it contains every measure Q with $Q[\,A\,] \le \gamma(A)$ for all $A \in \mathcal{F}_T$.*

(c) *There exists a Polish topology on Ω such that $\mathcal{F}_T$ is the corresponding Borel field and $\mathcal{Q}$ is compact.*

Let us also comment on conditions (b) and (c) in Assumption 3.6. Condition (c) guarantees that γ is a capacity in the sense of Choquet [6]. Condition (b) implies that $\mathcal{Q}$ is convex and closed in total variation. Hence, Lemma 2.4 yields that any least favorable measure must be equivalent to P^*. Moreover, under assumption (a), the set $\mathcal{Q} = \{Q \,|\, Q \leq \gamma\}$ is equal to

$$\Big\{ Q \,\Big|\; E_Q[\,X\,] \leq \int_0^\infty \gamma(X > t)\,dt \text{ for all } X \in L_+^\infty \Big\}$$

see Section 5 of [8].

Consider the 2-alternating set function

$$\nu_t(A) := t\gamma(A) - P^*[\,A\,]\,, \qquad A \in \mathcal{F}_T\,. \tag{8}$$

It is shown in Lemmas 3.1 and 3.2 of [17] that under Assumption 3.6 there exists a decreasing family $(A_t)_{t>0} \subset \mathcal{F}_T$ such that A_t minimizes ν_t and such that $A_t = \bigcup_{s>t} A_s$.

Definition 3.7 (Huber and Strassen) *The function*

$$\frac{dP^*}{d\gamma}(\omega) = \inf\{\, t \,|\, \omega \notin A_t \,\}\,, \qquad \omega \in \Omega\,,$$

is called the Radon-Nikodym derivative *of P^* with respect to γ.*

The terminology “Radon-Nikodym derivative” comes from the fact that $dP^*/d\gamma$ coincides with the usual Radon-Nikodym derivative dP^*/dQ in case where $\mathcal{Q} = \{Q\}$; see [17]. We will need the following simple lemma:

Lemma 3.8 *Condition* (4) *implies that* $P\big[\,0 < \frac{dP^*}{d\gamma} < \infty\,\big] = 1$.

Proof: Let ν_t be as in (8). Clearly, $\frac{dP^*}{d\gamma}(\omega) = \infty$ if and only if $\omega \in A_\infty := \bigcap_{0<t<\infty} A_t$. Since $\nu_t(A_t) \leq \nu_t(\emptyset) = 0$, we have $\gamma(A_t) \leq 1/t$. It follows that $\gamma(A_\infty) = 0$, which by (4) implies that $P[\,A_\infty\,] = 0$.

Letting $A_0 := \bigcup_{0<t<\infty} A_t$, we see that $\frac{dP^*}{d\gamma}(\omega) = 0$ if and only if $\omega \in A_0^c$. From $\nu_t(A_t) \leq \nu_t(\Omega) = t - 1$, we find that $P^*[\,A_t^c\,] \leq t(1 - \gamma(A_t))$. As $t \downarrow 0$ we thus get $P^*[\,A_0^c\,] = 0$. $\square$

Let us now state the Huber-Strassen theorem from [17] in a form in which it will be needed here.

Theorem 3.9 (Huber-Strassen) *Under Assumption 3.6, $\mathcal{Q}$ admits a least favorable measure Q_0 with respect to any probability measure R on $(\Omega, \mathcal{F}_T)$. Moreover, if $R = P^*$ and $\mathcal{Q}$ satisfies* (4)*, then Q_0 is equivalent to P^* and given by*

$$dQ_0 = \Big(\frac{dP^*}{d\gamma}\Big)^{-1} dP^*\,.$$

Together with Theorem 2.2, we get a complete solution of the robust utility maximization problem within the large class of utility functionals that arise from sets $\mathcal{Q}$ as in Assumption 3.6. Before discussing particular examples, let us state the following converse of the Huber-Strassen theorem in order to clarify the role of condition (a) in Assumption 3.6.

Theorem 3.10 *Suppose Ω is a Polish space with Borel field $\mathcal{F}_T$ and $\mathcal{Q}$ is a compact set of probability measures. If every probability measure on $(\Omega, \mathcal{F}_T)$ admits a least favorable measure $Q_0 \in \mathcal{Q}$, then $\gamma(A) = \sup_{Q\in\mathcal{Q}} Q[\,A\,]$ is 2-alternating.*

For finite probability spaces, Theorem 3.10 is due to Huber and Strassen [17]. In the form stated above, it was proved by Lembcke [23]. An alternative formulation was given earlier by Bednarski [3].

Let us now turn to the discussion of examples. The following example class was first studied by Bednarski [2] under slightly different conditions than here. These examples also play a role in the theory of law-invariant risk measures; see Kusuoka [22] and Sections 4.4 through 4.7 in [10].

Example 3.11 The following class of 2-alternating set functions arises in Yaari's [30] "dual theory of choice under risk". Let $\psi : [0,1] \to [0,1]$ be an increasing concave function with $\psi(0) = 0$ and $\psi(1) = 1$. In particular, ψ is continuous on $(0,1]$. We define γ by

$$\gamma(A) := \psi\big(P[\,A\,]\big)\,, \qquad A \in \mathcal{F}\,.$$

Then γ is 2-alternating, and the set $\mathcal{Q}$ of all probability measures Q on $(\Omega, \mathcal{F}_T)$ with $Q[\,A\,] \le \gamma(A)$ can be described in terms of ψ:

$$\mathcal{Q} = \Big\{\, Q \ll P \,\Big|\, \varphi := \frac{dQ}{dP} \text{ satisfies } \int_t^1 q_\varphi(s)\,ds \le \psi(1-t) \text{ for } t \in (0,1) \,\Big\};$$

see Theorem 4.73 in [10]. If $(\Omega, \mathcal{F}_T)$ is a standard Borel space, then there exists a compact metric topology on Ω whose Borel field is $\mathcal{F}_T$. For such a topology, $\mathcal{Q}$ is weakly compact, and so Assumption 3.6 is satisfied and $\mathcal{Q}$ admits a least favorable measure Q_0. It can be explicitly determined in the case in which $\psi(t) = (t\lambda^{-1}) \wedge 1$ for some $\lambda \in (0,1)$. Before we describe it, let us note first that the corresponding set $\mathcal{Q}$ is given by

$$\mathcal{Q} = \Big\{\, Q \ll P \,\Big|\, \frac{dQ}{dP} \le \frac{1}{\lambda} \,\Big\};$$

see, e.g., [10], Section 4.6. Next, suppose for simplicity that $\varphi := dP^*/dP$ has a continuous and strictly increasing distribution function F_φ under P, and denote by q_φ the corresponding quantile function (i.e., the generalized inverse of F_φ). Then the function

$$(0,1] \ni y \longmapsto \frac{y + \lambda - 1}{\int_0^y q_\varphi(t)\,dt}$$

has a unique maximizer $y_\lambda \in (1-\lambda, 1]$, and the Radon-Nikodym derivative of P^* with respect to γ is given by

$$\pi = \frac{dP^*}{d\gamma} = \lambda \cdot (\varphi \vee q_\varphi(y_\lambda)) \,,$$

as is proved in [26, Remark 4.6]. If $\|\varphi\|_{L^\infty} > \lambda^{-1}$, then y_λ is the unique solution of the equation

$$q_\varphi(y)(y+\lambda-1) = \int_0^y q_\varphi(t)\, dt \,.$$

Apart from this special case, an explicit formula for $\pi = dP^*/d\gamma$ is not known to the writer, but π can be computed (in principle and numerically) by solving a certain non-linear variational problem in two real parameters; see Section 4 of [26]. ◇

Example 3.12 (Weak information) Let Y be a measurable function on $(\Omega, \mathcal{F}_T)$, and denote by μ its law under P^*. For $\nu \approx \mu$ given, let

$$\mathcal{Q} := \Big\{\, Q \ll P^* \,\Big|\, Q \circ Y^{-1} = \nu \,\Big\} \,.$$

The robust utility maximization problem for this set $\mathcal{Q}$ was studied by Baudoin [4], who coined the terminology "weak information". The interpretation behind the set $\mathcal{Q}$ is that an investor has full knowledge about the pricing measure P^* but is uncertain about the true distribution P of market prices and only knows that a certain functional Y of the stock price has distribution ν; see Example 3.14 below for an extension where the investor has only partial knowledge about the distribution of Y. The set $\mathcal{Q}$ can also be regarded as the class of all extensions of $P|_{\sigma(Y)}$ to the full σ-field $\mathcal{F}_T$; this point of view is taken by Plachky and Rüschendorf in "Conservation of the UMP-resp. maximin-property of statistical tests under extensions of probability measures", Goodness-of-fit, Debrecen/Hung. 1984, Colloq. Math. Soc. Janos Bolyai 45, 439-457 (1987).

Define Q_0 by

$$dQ_0 = \frac{d\mu}{d\nu}(Y)\, dP^* .$$

Then $Q_0 \in \mathcal{Q}$ and the law of $\pi := dQ_0/dP^* = d\mu/d\nu(Y)$ is the same for all $Q \in \mathcal{Q}$. In particular, $Q_0[\,\pi \le t\,] = \inf_{Q\in\mathcal{Q}} Q[\,\pi \le t\,]$, and so Q_0 is a least favorable measure. The same procedure can be applied to *any* measure $R \approx P^*$. In fact, $\mathcal{Q}$ fits into the framework of the Huber-Strassen theory, as is shown in the following proposition. ◇

Proposition 3.13 *Suppose $(\Omega, \mathcal{F}_T)$ is a standard Borel space. Then the set $\mathcal{Q}$ defined in Example 3.12 satisfies Assumption 3.6. In particular, $\gamma(A) := \sup_{Q\in\mathcal{Q}} Q[\,A\,]$ is 2-alternating.*

Proof: If Q is a probability measure with $Q[\,\cdot\,] \le \gamma(\,\cdot\,)$, then

$$Q[\,Y \le t\,] \le \gamma(Y \le t) = \nu((-\infty, t]) \,.$$

Using the same argument on $\{Y > t\}$ shows that Y has law ν under Q. Hence, $\mathcal{Q}$ is maximal in the sense of part (b) of Assumption 3.6.

To this end, we may choose a compact metric topology on Ω such that Y is continuous and $\mathcal{F}_T$ is the Borel σ-algebra. Then $\mathcal{Q}$ is weakly compact, and condition (c) is satisfied.

To prove that part (a) holds we will use Theorem 3.10. The weak compactness assumption in this theorem is satisfied by the preceding argument. To show that any measure R admits a least favorable measure, write $P^* = \mu K^* := \int \mu(dy)K^*(y,\cdot)$, where $K^*(y,\cdot) = P^*[\,\cdot\,|\,Y = y\,]$ is a regular conditional expectation given Y. If $R \ll P^*$, then $\eta := R \circ Y^{-1} \ll \nu$ and R can be written as ηK_R, where K_R is a stochastic kernel such that $K_R(y,\cdot) \ll K^*(y,\cdot)$ for η-a.e. y. Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν with respect to η into the absolutely continuous part $\nu_a \ll \eta$ and into the singular part ν_s. If we let $Q_0 := \nu_a K_R + \nu_s K^*$, then $Q_0 \in \mathcal{Q}$ and

$$\pi = \frac{dR}{dQ_0} = \frac{d\eta}{d\nu}(Y)\,.$$

Again, the distribution of π is the same for all $Q \in \mathcal{Q}$, and it follows that Q_0 is a least favorable measure. If $R \not\ll P^*$, then it is clear that any measure Q_0 will be least favorable for R if it is least favorable for the absolutely continuous part of R. □

In the 1970's and 1980's, explicit formulas for Radon-Nikodym derivatives with respect to capacities were found in a number of examples such as sets $\mathcal{Q}$ defined in terms of ε-contamination or via probability metrics like total variation or Prohorov distance; we refer to Chapter 10 in the book [16] by Huber and the references therein. But, unless Ω is finite, these examples fail to satisfy either implication in (4) (see, however, Example 3.14 below). Nevertheless, they are still interesting for discrete-time market models.

3.3 Further examples from robust statistics

In this section, we briefly discuss further example classes that may or may not lead to 2-alternating capacities but for which least favorable measures are available.

Example 3.14 (Huber [16]) Let Y be a real-valued random with distributions μ and μ^* under P and P^*, respectively. Suppose that $d\mu^*/d\mu$ is an increasing function on the real line. For $\varepsilon, \delta \in [0, 1)$, we define

$$\mathcal{Q} := \big\{\, Q \ll P \,\big|\, Q[\,Y < t\,] \geq (1-\varepsilon)P[\,Y < t\,] - \delta \text{ for all } t \,\big\}\,.$$

This class of examples includes ε-contamination and neighborhoods of $P \circ Y^{-1}$ with respect to the following probability metrics: total variation, Prohorov metric, Kolmogorov distance, and Lévy metric; see [16], p. 271. The financial interpretation is similar to the case of "weak information" in Example 3.12: The investor only has knowledge about the distribution of Y, but now this knowledge is itself subject to uncertainty. Under the above conditions, one can show that $\mathcal{Q}$ admits a least favorable measure Q_0, and π is proportional to $c' \vee d\mu^*/d\mu(X) \wedge c''$ for certain constants c' and c''. We refer to Section 10.3 of [16] for details. ◇

Example 3.15 (Augustin [1]) Here one starts with any set $\mathcal{Q}$ that admits an equivalent least favorable measure Q_0 and applies a distortion function ψ to the upper probability arising from $\mathcal{Q}$:

$$\overline{\mathcal{Q}} := \Big\{ \overline{Q} \,\Big|\, \overline{Q}[\,A\,] \le \psi\Big(\sup_{Q\in\mathcal{Q}} Q[\,A\,]\Big) \text{ for all } A \in \mathcal{F}_T \Big\}\,.$$

Here, $\psi : [0,1] \to [0,1]$ is increasing and concave with $\psi(0) = 0$ and $\psi(1) = 1$ as in Example 3.11. The need for considering sets $\overline{\mathcal{Q}}$ of this form might arise if $\mathcal{Q}$ itself does not fully capture the uncertainty of the situation so that it needs further enlargement. That is, the set $\mathcal{Q}$ is itself subject to uncertainty. Augustin [1] gives various conditions under which a least favorable measure $\overline{Q}_0$ for the σ-core of the 2-alternating set function $\psi(Q_0[\,\cdot\,])$ is also a least favorable measure for $\overline{\mathcal{Q}}$. $\diamond$

4 Proofs of the Theorems 2.2 and 2.5

Let X^* be a solution of the standard utility maximization problem for the least favorable measure Q_0. Then it is well known that $X^*_T = I(y\pi)$ for some constant $y > 0$. Thus, one easily checks via Proposition 3.1 that X^* is also a solution of the robust utility maximization problem. However, in order to show the full equivalence of the two problems, we must also take care of the situation in which the standard problem has no solution. Our key result is the following proposition.

Proposition 4.1 *Let $Q_0 \approx P^*$ be a least favorable measure and $\pi = dP^*/dQ_0$.*

(a) *For any $X \in \mathcal{X}(x)$ there exists $\widetilde{X} \in \mathcal{X}(x)$ such that*

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(\widetilde{X}_T)\,] \ge \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,]$$

and such that $\widetilde{X}_T = f(\pi)$ for some deterministic decreasing function $f : (0,\infty) \to [0,\infty)$.

(b) *The terminal wealth of any solution X^* of (3) is of the form $X^*_T = f^*(\pi)$ for a deterministic decreasing function $f^*(0,\infty) \to [0,\infty)$.*

The proof of this proposition is based on ideas from [26] and on the following version of the classical Hardy-Littlewood inequalities, which we recall here for the convenience of the reader. See, e.g., [10, Theorem A.24] for a proof.

Theorem 4.2 (Hardy-Littlewood) *Let X and Y be two non-negative random variables on $(\Omega, \mathcal{F}_T, Q)$, and let q_X and q_Y denote quantile functions of X and Y with respect to Q. Then,*

$$\int_0^1 q_X(1-t)q_Y(t)\,dt \le E_Q[\,XY\,] \le \int_0^1 q_X(t)q_Y(t)\,dt\,.$$

If $X = f(Y)$, then the lower (upper) bound is attained if and only if f can be chosen as a decreasing (increasing) function.

Proof of Proposition 4.1: (a) By market completeness, it suffices to construct a decreasing function $f \geq 0$ such that $E^*[\,f(\pi)\,] \leq x$ and

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(f(\pi))\,] \geq \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,]\,. \tag{9}$$

To this end, we denote by $F_Y(x) := Q_0[\,Y \leq x\,]$ the distribution function and by $q_Y(t)$ a quantile function of a random variable Y with respect to the probability measure Q_0. We will need the the following basic property of quantile functions: If f is a decreasing or increasing function and $Y \geq 0$, then any quantile function $q_{f(Y)}$ of $f(Y)$ satisfies for a.e. $t \in (0,1)$

$$q_{f(Y)}(t) = \begin{cases} f(q_Y(1-t)) & \text{if } f \text{ is decreasing,} \\ f(q_Y(t)) & \text{if } f \text{ is increasing;} \end{cases} \tag{10}$$

see, e.g., [10, Lemma A.23].

Let us define a function f by

$$f(t) := \begin{cases} q_{X_T}(1 - F_\pi(t)) & \text{if } F_\pi \text{ is continuous at } t, \\ \\ \dfrac{1}{F_\pi(t) - F_\pi(t-)} \displaystyle\int_{F_\pi(t-)}^{F_\pi(t)} q_{X_T}(1-s)\,ds & \text{otherwise.} \end{cases} \tag{11}$$

Then f is decreasing and satisfies $f(q_\pi) = E_\lambda[\,h \,|\, q_\pi\,]$, where λ is the Lebesgue measure and $h(t) := q_{X_T}(1-t)$. Hence, Jensen's inequality for conditional expectations and (10) show that

$$\begin{aligned} \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,] &\leq E_{Q_0}[\,U(X_T)\,] = \int_0^1 U(h(t))\,dt \\ &\leq \int_0^1 U\big(E_\lambda[\,h \,|\, q_\pi\,](t)\big)\,dt = \int_0^1 U(q_{f(\pi)}(1-t))\,dt \\ &= E_{Q_0}[\,U(f(\pi))\,] = \inf_{Q\in\mathcal{Q}} E_Q[\,U(f(\pi))\,]\,, \end{aligned} \tag{12}$$

where we have used Proposition 3.1 in the last step. Thus, f satisfies (9).

It remains to show that $f(\pi)$ satisfies the capital constraint. To this end, we first use the lower Hardy-Littlewood inequality:

$$x \geq E^*[\,X_T\,] = E_{Q_0}[\,\pi X_T\,] \geq \int_0^1 q_\pi(t)\,q_{X_T}(1-t)\,dt\,. \tag{13}$$

Here we may replace $q_{X_T}(1-t) = h(t)$ by $E_\lambda[\,h \,|\, q_\pi\,](t) = f(q_\pi(t))$. We then get

$$\int_0^1 q_\pi(t)\,q_{X_T}(1-t)\,dt = \int_0^1 q_\pi(t)\,f(q_\pi(t))\,dt = E_{Q_0}[\,\pi f(\pi)\,] = E^*[\,f(\pi)\,]\,. \tag{14}$$

Thus, f is as desired.

(b) Now suppose X^* solves (3). If X_T^* is not Q_0-a.s. $\sigma(\pi)$-measurable, then $Y := E_{Q_0}[\,X_T^*\,|\,\pi\,]$ must satisfy

$$E_{Q_0}[\,U(Y)\,] > E_{Q_0}[\,U(X_T^*)\,]\,, \tag{15}$$

due to the strict concavity of U. If we define $\widetilde{f}$ as in (11) with Y replacing X_T, then the proof of part (a) yields that

$$E^*[\,\widetilde{f}(\pi)\,] = E_{Q_0}[\,\pi\widetilde{f}(\pi)\,] \le E_{Q_0}[\,\pi Y\,] = E_{Q_0}[\,\pi X_T^*\,] \le x\,,$$

and by (12) and (15),

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U(\widetilde{f}(\pi))\,] \ge E_{Q_0}[\,U(Y)\,] > E_{Q_0}[\,U(X_T^*)\,] \ge \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T^*)\,]\,,$$

in contradiction to the optimality of X^*. Thus, X_T^* is necessarily $\sigma(\pi)$-measurable and can hence be written as a (not yet necessarily decreasing) function of π.

If we define f^* as in (11) with X_T^* replacing X_T, then $f^*(\pi)$ is the terminal wealth of yet another solution in $\mathcal{X}(x)$. Clearly, we must have $E^*[\,X_T^*\,] = x = E^*[\,f^*(\pi)\,]$. Thus, (13) and (14) yield that $E_{Q_0}[\,\pi X_T^*\,] = \int_0^1 q_\pi(t)\, q_{X_T^*}(1-t)\,dt$. But then the "only if" part of the lower Hardy-Littlewood inequality together with the $\sigma(\pi)$-measurability of X_T^* imply that X_T^* is a decreasing function of π. $\square$

Proof of Theorem 2.2: Proposition 4.1 implies that in solving the robust utility maximization problem (3) we may restrict ourselves to strategies whose terminal wealth is a decreasing function of π. By Propositions 3.1, the robust utility of a such a terminal wealth is the same as the expected utility with respect to Q_0. On the other hand, taking $\mathcal{Q}_0 := \{Q_0\}$ in Proposition 4.1 implies that the standard utility maximization problem for Q_0 also requires only strategies whose terminal wealth is a decreasing function of π. Therefore, the two problems are equivalent, and Theorem 2.2 is proved. $\square$

Proof of Theorem 2.5 Let (U_n) be a sequence of nonnegative and continuously differentiable utility functions that increase uniformly to the concave increasing function $U(x) := x \wedge 1$. Uniform convergence of U_n implies convergence of the corresponding value functions:

$$u_0^n(x) := \sup_{X\in\mathcal{X}(x)} E_{Q_0}[\,U_n(X_T)\,] \nearrow \sup_{X\in\mathcal{X}(x)} E_{Q_0}[\,U(X_T)\,] =: u_0(x). \tag{16}$$

If we assume that $U_1'(x)$ decreases fast enough to 0 as $x \uparrow \infty$, then $E^*[\,I_1^+(c\pi)\,] < \infty$ for all $c > 0$, where $\pi := dP^*/dQ_0$ and I_1^+ is the inverse of U_1' on $(0, U_1'(0))$ and $I_1^+(x) = 0$ for $x \ge U_1'(0)$. Market completeness and [10, Theorem 3.39] guarantee that, for every $0 < x \le 1$ and each $n \in \mathbb{N}$, there exists a solution $X^n \in \mathcal{X}(x)$ for the standard utility maximization problem with utility function U_n under Q_0. Note that the preceding two statements also remain true for $P^* \not\ll Q_0$, in which case $X_T^n = 0$ on $\{\pi = \infty\}$.

By a Komlos-type argument (see [20, Lemma 3.3]), there exists a sequence $Y_n \in \operatorname{conv}\{X_T^n, X_T^{n+1}, \dots\}$ which converges P^*-a.s. to some random variable $X_T^* \ge 0$, which satisfies $E^*[\,X_T^*\,] \le x$ due to Fatou's lemma. Hence, X_T^* corresponds to a value process $X^* \in \mathcal{X}(x)$. Let us write Y_n as the convex combination $Y_n = \sum_{k\ge n} \alpha_{k,n} X_T^k$, where only finitely many $\alpha_{k,n}$ are nonzero. Then,

$$\begin{aligned} u_0(x) &\ge E_{Q_0}[\,U(X_T^*)\,] = \lim_{n\uparrow\infty} E_{Q_0}[\,U(Y_n)\,] \ge \limsup_{n\uparrow\infty} \sum_{k\ge n} \alpha_{k,n} E_{Q_0}[\,U_k(X_T^k)\,] \\ &= \limsup_{n\uparrow\infty} \sum_{k\ge n} \alpha_{k,n} u_0^k(x) = u_0(x), \end{aligned}$$

due to (16). Hence, X^* is optimal for the utility maximization problem with U and Q_0. Since U is constant on $[1,\infty)$, we must have $0 \le X_T^* \le 1$ P^*-almost surely. Thus, X_T^* is a solution to the problem of maximizing $E_{Q_0}[\,U(X)\,] = E_{Q_0}[\,X\,]$ under the constraints $0 \le X \le 1$ and $E^*[\,X\,] \le x$. Hence, the generalized Neyman-Pearson lemma in the form of [10, Theorem A.30] implies that $X_T^* = \mathrm{I}_{\{\pi<q\}} + \kappa \mathrm{I}_{\{\pi=q\}}$, where q can be any x-quantile for the law of π under P^*, and κ is a $[0,1]$-valued random variable. In particular,

$$X_T^* = \mathrm{I}_{\{\pi\le q\}} \qquad P^*\text{-a.s. for } x \text{ with } P^*[\,\pi = q\,] = 0. \tag{17}$$

Note also that the x-quantile q is unique if $P^*[\,\pi = q] > 0$.

Next, if $Q \in \mathcal{Q}$ is given, then

$$\begin{aligned} E_Q[\,U(X_T^*)\,] &= \lim_{n\uparrow\infty} E_Q[\,U(Y_n)\,] \ge \limsup_{n\uparrow\infty} \sum_{k\ge n} \alpha_{k,n} E_Q[\,U_k(X_T^k)\,] \\ &\ge \limsup_{n\uparrow\infty} \sum_{k\ge n} \alpha_{k,n} E_{Q_0}[\,U_k(X_T^k)\,] = \limsup_{n\uparrow\infty} \sum_{k\ge n} \alpha_{k,n} u_0^k(x) \qquad (18) \\ &= u_0(x) = E_{Q_0}[\,U(X_T^*)\,], \end{aligned}$$

where we have used the fact that $E_Q[\,U_k(X_T^k)\,] \ge E_{Q_0}[\,U_k(X_T^k)\,]$ for all k. This inequality follows from the hypothesis of the theorem: X_T^k solves both the standard and the robust utility maximization problems, and the corresponding value functions are equal, i.e.,

$$\inf_{Q\in\mathcal{Q}} E_Q[\,U_k(X_T^k)\,] = \sup_{X\in\mathcal{X}(x)} \inf_{Q\in\mathcal{Q}} E_Q[\,U_k(X_T)\,] = \sup_{X\in\mathcal{X}(x)} E_{Q_0}[\,U_k(X_T)\,] = E_{Q_0}[\,U_k(X_T^k)\,].$$

Finally, combining (18) with (17) yields $Q[\pi \le q\,] = E_Q[\,U(X_T^*)\,] \ge E_{Q_0}[\,U(X_T^*)\,] = Q_0[\,\pi \le q\,]$ for all but countably many and, in turn, all $q \in [0,1]$. □

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References

[1] T. Augustin, *Neyman-Pearson testing under interval probability by globally least favorable pairs reviewing Huber-Strassen theory and extending it to general interval probability,* Imprecise probability models and their applications (Ghent, 1999). J. Statist. Plann. Inference **105** (2002) 149–173.

[2] T. Bednarski, *On solutions of minimax test problems for special capacities,* Z. Wahrsch. Verw. Gebiete **58** (1981) 397–405.

[3] T. Bednarski, *Binary experiments, minimax tests and* 2*-alternating capacities,* Ann. Statist. **10** (1982) 226–232.

[4] F. Baudoin, *Conditioned stochastic differential equations: theory, examples and application to finance,* Stochastic Process. Appl. **100** (2002) 109–145.

[5] Z. Chen and A. Sulem, *An integral representation theorem of g-expectations,* Inria rapport de recherche no. 4284, 2001.

[6] G. Choquet, *Theory of capacities,* Ann. Inst. Fourier **5** (1953/1954) 131-292.

[7] R. Cont, *Model uncertainty and its impact on the pricing of derivative instruments,* Preprint, École Polytechnique de Paris, 2004.

[8] F. Delbaen, *Coherent risk measures,* Lecture notes, Scuola Normale di Pisa, 2000.

[9] N. El Karoui, M. Jeanblanc-Picqué, and S. Shreve, *Robustness of the Black and Scholes formula,* Math. Finance **8** (1998) 93–126.

[10] H. Föllmer and A. Schied, *Stochastic Finance: An Introduction in Discrete Time,* 2nd edition. Berlin: de Gruyter Studies in Mathematics 27, 2004.

[11] I. Gilboa, *Expected utility with purely subjective non-additive probabilities,* J. Math. Econ. **16** (1987) 65-88.

[12] I. Gilboa and D. Schmeidler, *Maxmin expected utility with non-unique prior,* J. Math. Econ. **18** (1989) 141-153.

[13] A. Gundel, *Robust utility maximization in complete and incomplete market models,* Preprint, Humboldt-Universität zu Berlin, 2003.

[14] B. Hajek, Mean stochastic comparison of diffusions, Z. Wahrscheinlichkeitstheor. Verw. Geb., **68**, (1985) 315-329.

[15] D. Hobson, *Volatility mis-specification, option pricing and super-replication via coupling,* Ann. Appl. Probab. **8** (1998) 193-205.

[16] P. Huber, *Robust statistics,* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1981.

[17] P. Huber and V. Strassen, *Minimax tests and the Neyman-Pearson lemma for capacities,* Ann. Statistics **1** (1973) 251-263. *Correction:* Ann. Statistics **2** (1974) 223-224.

[18] S. Janson and J. Tysk, *Volatility time and properties of option prices,* Ann. Appl. Probab. **13** (2003) 890-913.

[19] I. Karatzas and S. Shreve, *Methods of mathematical finance,* Applications of Mathematics. Berlin: Springer, 1998.

[20] D. Kramkov and W. Schachermayer, *The asymptotic elasticity of utility functions and optimal investment in incomplete markets,* Ann. Appl. Probab. **9** (1999) no. 3, 904–950.

[21] D. Kramkov and W. Schachermayer, *Necessary and sufficient conditions in the problem of optimal investment in incomplete markets,* Ann. Appl. Probab., Vol.**13** (2003) 1504-1516.

[22] S. Kusuoka, *On law invariant coherent risk measures,* Adv. Math. Econ. **3** (2001) 83-95.

[23] J. Lembcke, *The necessity of strongly subadditive capacities for Neyman-Pearson minimax tests,* Monatsh. Math. **105** (1988) 113–126.

[24] S. Peng, *Backward SDE and related g-expectation,* In: Backward stochastic differential equations (Paris, 1995–1996), Pitman Res. Notes Math. Ser., 364, Longman, Harlow, 1997, pp. 141–159.

[25] W. Schachermayer, *Optimal investment in incomplete financial markets,* Proceedings of the first World Congress of the Bachelier Society, Paris 2000, Springer-Verlag, Berlin, 2001.

[26] A. Schied, *On the Neyman-Pearson problem for law-invariant risk measures and robust utility functionals,* Ann. Appl. Probab. **14** (2004) 13981423.

[27] A. Schied and C.-T. Wu, *Duality theory for robust utility maximization in incomplete market models,* Preprint TU Berlin, 2004.

[28] D. Schmeidler, *Subjective probability and expected utility without additivity,* Econometrica bf 57 (1989) 571-587.

[29] C. Skiadas, *Robust control and recursive utility,* Finance Stoch. **7** (2003) 475–489 .

[30] M. Yaari, *The dual theory of choice under risk,* Econometrica **55** (1987) 95-115.