Robust optimal control for a consumption-investment problem

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Abstract: We give an explicit PDE characterization for the solution of the problem of maximizing the utility of both terminal wealth and intertemporal consumption under model uncertainty. The underlying market model consists of a risky asset, whose volatility and long-term trend are driven by an external stochastic factor process. The robust utility functional is defined in terms of a HARA utility function with risk aversion parameter $0 < \alpha < 1$ and a dynamically consistent coherent risk measure, which allows for model uncertainty in the distributions of both the asset price dynamics and the factor process. Our method combines recent results by Wittmüss (2007) on the duality theory of robust optimization of consumption with a stochastic control approach to the dual problem of determining a 'worst-case martingale measure'.

1 Introduction

Recently, there has been considerable interest in studying optimization problems in which the target functional is defined in terms of a coherent or convex risk measure. These optimization problems can be called *robust* since optimization involves an entire class $\mathcal{Q}$ of possible probabilistic models and thus takes into account model risk; see, e.g., [24] and the references therein. This link between model uncertainty and risk measures is particularly transparent in the theory of investors preferences under model uncertainty as developed by Gilboa and Schmeidler [12]. By introducing an axiom called 'uncertainty aversion' within an extended von Neumann-Morgenstern framework, Gilboa and Schmeidler [12] derive the following representation for the corresponding utility functional:

$$X \longmapsto \inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,],$$

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where $\mathcal{Q}$ is a set of probability measures, and U is a utility function. A natural question is now to study some of classical problems of mathematical finance and economics within this setup. Optimal investment problems for such robust utility functionals were considered, among others, by Talay and Zheng [27], Korn and Wilmott [19], Quenez [22], Schied [23], Korn and Menkens [17], Gundel [13], Schied and Wu [26], Föllmer and Gundel [8], Korn and Steffensen [18], and Hernández-Hernández and Schied [14, 15].

The present paper is a continuation of [14], where the problem of maximizing the robust utility of the terminal wealth was studied in a stochastic factor model and for HARA utility functions

$$U(x) = \frac{x^\alpha}{\alpha}, \qquad x > 0,$$

with risk aversion parameter $\alpha < 0$. Here, we will discuss the case $\alpha > 0$, which is more difficult than the case $\alpha < 0$ and requires completely different methods. We will moreover allow for intertemporal consumption strategies, which is important for several fascinating applications in macro-economic theory; see, e.g., Barillas et al. [1] and the references therein. Also the setup of our market model is more general than in [14] and now includes local volatility models.

Our method relies first on an application of the duality results for the robust optimization of consumption obtained by Wittmüss [28] (earlier results on the same problem were obtained by Burgert and Rüschendorf [2], but they are not applicable to our situation, due to more restrictive assumptions). The idea of using convex duality so as to transform the original minimax problem into a minimization problem was first used by Quenez [22]. After using [28] to set up the dual problem as a two-parameter minimization problem, we then use stochastic control techniques to derive a Hamilton-Jacobi-Bellman equation for the value function v. Our main result states that v is in fact a classical solution of this quasi-linear PDE. In particular, we avoid the use of (non-smooth) viscosity solutions and thus obtain explicit formulas for the optimal strategy in terms of v and its derivatives.

The increased difficulty of the problem for $\alpha > 0$ in comparison to the case $\alpha < 0$ is related to the fact that a 'worst-case martingale measure' may not exist and that the infimum may only be attained within a larger class of sub-probability measures. This phenomenon is well-known also in standard utility maximization; see Kramkov and Schachermayer [20, Section 5]. On the analytical side, it corresponds to the possible unboundedness of the gradient of the value function v in the case $\alpha > 0$; see Lemma 3.5 and its proof. Establishing the boundedness of this gradient in the case $\alpha < 0$ was the key step in [14].

The paper is organized as follows. In the next section, we introduce our model and state our main result. Its proof is given in Section 3.

2 Statement of main results

We consider a financial market model with a locally riskless money market account

$$dS_t^0 = S_t^0 r(Y_t)\, dt \tag{1}$$

with locally risk-free rate $r \geq 0$ and a risky asset defined under a reference measure $\mathbb{P}$ through the SDE

$$dS_t = S_t b(Y_t)\,dt + S_t \sigma(Y_t)\,dW_t^1. \tag{2}$$

Here W^1 is a standard $\mathbb{P}$-Brownian motion and Y denotes an external economic factor process modeled by the SDE

$$dY_t = g(Y_t)\,dt + \rho(Y_t)\,dW_t^1 + \varsigma(Y_t)\,dW_t^2 \tag{3}$$

for a standard $\mathbb{P}$-Brownian motion W^2, which is independent of W^1 under $\mathbb{P}$. We suppose that the economic factor can be observed but cannot be traded directly so that the market model is typically incomplete. Models of this type have been widely used in finance and economics, the case of a mean-reverting factor process with the choice $g(y) := -\kappa(\mu - y)$ being particularly popular; see, e.g., Fleming and Hernández-Hernández [4], Fouque et al. [10], and the references therein. We assume that g belongs to $C^2(\mathbb{R})$, with derivative $g' \in C_b^1(\mathbb{R})$, and r, b, σ, ρ, and ς belong to $C_b^2(\mathbb{R})$, where $C_b^k(\mathbb{R})$ denotes the class of bounded functions with bounded derivatives up to order k. We will also assume that

$$\sigma(y) \geq \sigma_0 \text{ and } a(y) := \frac{1}{2}(\rho^2(y) + \varsigma^2(y)) \geq \sigma_1^2 \text{ for some constants } \sigma_0, \sigma_1 > 0. \tag{4}$$

The market price of risk with respect to the reference measure $\mathbb{P}$ is defined via the function

$$\theta(y) := \frac{b(y) - r(y)}{\sigma(y)}.$$

The assumption of time-independent coefficients is for convenience in the exposition only and can be relaxed by standard arguments. Similarly, it is easy to extend our results to a d-dimensional stock market model replacing the one-dimensional SDE (2).

Remark 2.1 By taking $\varsigma \equiv 0$, $\rho(y) = \sigma(y)$, $g(y) = b(y) - \frac{1}{2}\sigma^2(y)$, and $Y_0 = \log S_0$ it follows that Y coincides with $\log S$. Hence, S solves the SDE of a local volatility model:

$$dS_t = S_t \widetilde{b}(S_t)\,dt + S_t \widetilde{\sigma}(S_t)\,dW_t^1, \tag{5}$$

where $\widetilde{b}(x) = b(\log x)$ and $\widetilde{\sigma}(x) = \sigma(\log x)$. Thus, our analysis includes the study of the robust optimal investment problem for local volatility models given by (5), and it will be easy to derive the corresponding equation as a special case of our main result, Theorem 2.2.

In most economic situations, investors typically face *model uncertainty* in the sense that the dynamics of the relevant quantities are not precisely known. One common approach to coping with model uncertainty is to admit an entire class $\mathcal{Q}$ of possible prior models. Here, we will consider the class

$$\mathcal{Q} := \Big\{ Q \sim \mathbb{P} \,\Big|\, \frac{dQ}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta_{1t}\,dW_t^1 + \int_0 \eta_{2t}\,dW_t^2 \Big)_T,\ \eta = (\eta_1, \eta_2) \in \mathcal{C} \Big\},$$

where $\mathcal{E}(M)_t = \exp(M_t - \langle M\rangle_t/2)$ denotes the Doleans-Dade exponential of a continuous local martingale M and $\mathcal{C}$ denotes the set of all progressively measurable processes $\eta = (\eta_1, \eta_2)$ such that η_t belongs $dt \otimes d\mathbb{P}$-a.e. to some fixed compact convex set $\Gamma \subset \mathbb{R}^2$. Note that due to Novikov's theorem we have a one-to-one correspondence between measures $Q \in \mathcal{Q}$ and processes $\eta \in \mathcal{C}$ (up to $dt \otimes d\mathbb{P}$-nullsets).

Let $\mathcal{A}$ denote the set of all pairs (c, π) of progressively measurable process π and c such that $c \geq 0$, $\int_0^T c_s\, ds < \infty$, and $\int_0^T \pi_s^2\, ds < \infty$ $\mathbb{P}$-a.s. For $(c, \pi) \in \mathcal{A}$ we define $X^{x,c,\pi}$ as the unique solution of the linear SDE

$$dX_t^{x,c,\pi} = \frac{X_s^{x,c,\pi}\pi_s}{S_s}\, dS_s + \frac{X_s^{x,c,\pi}(1-\pi_s)}{S_s^0}\, dS_s^0 - c_s\, ds \qquad \text{and} \qquad X_0^{x,c,\pi} = x. \tag{6}$$

Then $X^{x,c,\pi}$ describes the evolution of the wealth process of an investor with initial endowment $X_0^{x,c,\pi} = x > 0$ who is consuming at the rate c_s and investing the fraction π_s of the current wealth $X_s^{x,c,\pi}$ into the risky asset at time $s \in [0, T]$. By $\mathcal{A}(x)$ we denote the subclass of all $(c, \pi) \in \mathcal{A}$ that are admissible in the sense that $X_t^{x,c,\pi} \geq 0$ $\mathbb{P}$-a.s. for all t.

The objective of the investor consists in

$$\text{maximizing} \quad \inf_{Q\in\mathcal{Q}} E_Q\Big[\int_0^T \gamma e^{-\lambda t} U(c_t)\, dt + U(X_T^{x,c,\pi})\Big] \quad \text{over} \quad (c, \pi) \in \mathcal{A}(x), \tag{7}$$

where $\gamma, \lambda \geq 0$, and the utility function $U :]0, \infty[\to \mathbb{R}$ will be specified in the sequel as a HARA utility function with risk aversion parameter $\alpha > 0$:

$$U(x) = \frac{x^\alpha}{\alpha}. \tag{8}$$

By taking $\gamma = 0$, we obtain as a special case the optimization problem for the terminal wealth:

$$\text{maximize} \quad \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T^{x,0,\pi})\,] \quad \text{over } \pi \text{ such that} \quad (0, \pi) \in \mathcal{A}(x).$$

For the case $\alpha < 0$, this problem was studied in [14], but the case $\alpha > 0$ requires completely different methods. Finally, recall that $a = \frac{1}{2}(\rho^2 + \varsigma^2)$ and let us define

$$\beta := \frac{\alpha}{1-\alpha}.$$

Theorem 2.2 *There exists a unique strictly positive and bounded solution $v \in C^{1,2}(]0, T] \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ of the quasilinear PDE*

$$\begin{aligned} v_t = \gamma e^{-\lambda(T-t)} + a v_{yy} + (g + \beta\rho\theta) v_y - \frac{1}{2}\alpha\varsigma^2 \frac{v_y^2}{v} + \beta r v \\ + \inf_{\eta\in\Gamma} \Big[\big(\rho(1+\beta)\eta_1 + \beta\varsigma\eta_2\big) v_y + \frac{\beta(1+\beta)}{2}(\eta_1 + \theta)^2 v \Big] \end{aligned} \tag{9}$$

with initial condition

$$v(0, \cdot) \equiv 1, \tag{10}$$

and the value function of the robust utility maximization problem (7) *can then be expressed as*

$$u(x) := \sup_{(c,\pi)\in\mathcal{A}(x)} \inf_{Q\in\mathcal{Q}} E_Q\Big[\int_0^T \gamma e^{-\lambda t} U(c_t)\,dt + U(X_T^{x,c,\pi})\Big] = \frac{n_T^\alpha x^\alpha}{\alpha} v(T,Y_0)^{1-\alpha}, \qquad (11)$$

where $n_T := \frac{\gamma}{\lambda}(1-e^{-\lambda T})+1$. *If* $\eta^*(t,y)$ *is a measurable* Γ*-valued function that realizes the maximum in* (9)*, then an optimal strategy* $(\widehat{c},\widehat{\pi}) \in \mathcal{A}(x)$ *can be obtained by letting* $\widehat{\pi}_t = \pi^*(T-t,Y_t)$ *for*

$$\pi^*(t,y) = \frac{1}{\sigma(y)}\Big[(1+\beta)(\eta_1^*(t,y)+\theta(y)) + \rho(y)\frac{v_y(t,y)}{v(t,y)}\Big]$$

and by consuming at a rate proportional to the current total wealth $X_t^{x,\widehat{c},\widehat{\pi}}$*:*

$$\widehat{c}_t = \frac{\gamma e^{-\lambda t}}{v(T-t,Y_t)} X_t^{x,\widehat{c},\widehat{\pi}}.$$

Moreover, by defining a measure $\widehat{Q} \in \mathcal{Q}$ *via*

$$\frac{d\widehat{Q}}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta_1^*(T-t,Y_t)\,dW_t^1 + \int_0 \eta_2^*(T-t,Y_t)\,dW_t^2\Big)_T,$$

we obtain a saddlepoint $((\widehat{c},\widehat{\pi}),\widehat{Q})$ *for the maximin problem* (7).

Remark 2.3 For $\gamma = 0$ the HJB equation (9) can be simplified by passing to the log-transorm $w := \log v$; see [14].

3 Proof of the main result

We will first set up the dual problem to (7) following Wittmüss [28]. To check for the applicability of the results in [28], note first that our choice (8) obviously satisfies [28, Assumption 2.2]. Moreover, the convex risk measure

$$\rho(Y) := \sup_{Q\in\mathcal{Q}} E_Q[\,-Y\,], \qquad Y \in L^\infty(\mathbb{P}),$$

is continuous from below on $L^\infty(\mathbb{P})$. This follows by combining [14, Lemma 3.1], [26, Lemma 3.2], and [9, Corollary 4.35]. Hence, [28, Assumption 2.1] is also satisfied.

Let us denote by $\mathcal{M}$ the set of all progressively measurable processes ν such that $\int_0^T \nu_t^2\,dt < \infty$ $\mathbb{P}$-a.s., and define

$$Z_t^\nu := \mathcal{E}\Big(-\int \theta(Y_s)\,dW_s^1 - \int \nu_s\,dW_s^2\Big)_t.$$

Moreover, we introduce the conjugate function $\widetilde{U}(z) = \sup_{x\geq 0}(U(x)-zx)$ and the probability measure

$$\mu_T(dt) = \frac{1}{n_T}\big(\gamma e^{-\lambda t}\mathrm{I}_{[0,T]}(t)\,dt + \delta_T(dt)\big),$$

where n_T denotes the normalizing constant. It then follows from [28, Remark 2.7] and [16, Proposition 4.1] that, up to the normalizing constant n_T^{-1}, the dual value function of the robust utility maximization problem is given by

$$\widetilde{u}(z) := \inf_{\eta\in\mathcal{C}} \inf_{\nu\in\mathcal{M}} \mathbb{E}\Big[\int D_t^\eta \widetilde{U}(zZ_t^\nu/(D_t^\eta S_t^0))\, \mu_T(dt) \Big], \tag{12}$$

where

$$D_t^\eta = \mathcal{E}\Big(\int_0^{\cdot} \eta_s \, dW_s \Big)_t.$$

Due to [28, Theorem 2.5], the primal value function u can then be obtained as

$$u(x) = n_T \min_{z>0} (\widetilde{u}(z) + zx). \tag{13}$$

Moreover, the same result yields that if $\widehat{z} > 0$ minimizes (13) and there are control processes $(\widehat{\eta}, \widehat{\nu})$ minimizing (12) for $z = \widehat{z}$, then, for $I(y) := -\widetilde{U}'(y)$, the choice

$$\widehat{c}_t = \frac{1}{n_T} \gamma e^{-\lambda t} I\Big(\frac{\widehat{z} Z_t^{\widehat{\nu}}}{D_t^{\widehat{\eta}} S_t^0} \Big) \qquad \text{and} \qquad X_T^{x,\widehat{c},\widehat{\pi}} = \frac{1}{n_T} I\Big(\frac{\widehat{z} Z_T^{\widehat{\nu}}}{D_T^{\widehat{\eta}} S_T^0} \Big) \tag{14}$$

defines an optimal strategy $(\widehat{c}, \widehat{\pi}) \in \mathcal{A}(x)$. Here the factors $\gamma e^{-\lambda t}/n_T$ and $1/n_T$ come from the fact that in (6) we have introduced c as the consumption density with respect to the Lebesgue measure rather than with respect to μ_T as is required by [28]; $X_T^{x,c,\pi}$ plays the rol of a lump consumption at the terminal time T. In our specific setting (8), we have $\widetilde{U}(z) = z^{-\beta}/\beta$ with $\beta = \alpha/1-\alpha$. Thus, we can simplify the duality formula (13) as follows. First, the expectation in (12) equals

$$\mathbb{E}\Big[\int D_t^\eta \widetilde{U}\Big(\frac{zZ_t^\nu}{D_t^\eta S_t^0} \Big)\, \mu_T(dt) \Big] = \frac{z^{-\beta}}{\beta} \int \mathbb{E}\big[(D_t^\eta)^{1+\beta} (Z_t^\nu)^{-\beta} (S_t^0)^\beta \big]\, \mu_T(dt) =: \frac{z^{-\beta}}{\beta} \Lambda_{\eta,\nu}.$$

Optimizing over $z > 0$ then yields that

$$\min_{z>0} \Big(\frac{z^{-\beta}}{\beta} \Lambda_{\eta,\nu} + zx \Big) = \frac{1+\beta}{\beta} x^{\beta/(1+\beta)} \Lambda_{\eta,\nu}^{1/(1+\beta)} = \frac{x^\alpha}{\alpha} \Lambda_{\eta,\nu}^{1-\alpha},$$

where the optimal z is given by

$$\widehat{z} = \Big(\frac{\Lambda_{\eta,\nu}}{x} \Big)^{1/(1+\beta)} = \Big(\frac{\Lambda_{\eta,\nu}}{x} \Big)^{1-\alpha}. \tag{15}$$

Using (12) and (13) now yields

$$u(x) = n_T \frac{x^\alpha}{\alpha} \big(\inf_{\nu\in\mathcal{M}} \inf_{\eta\in\mathcal{C}} \Lambda_{\eta,\nu} \big)^{1-\alpha}. \tag{16}$$

By taking the strategy $(c, \pi) \equiv (x/(T+1), 0)$ in the definition (11) of u we obtain $u(x) \geq n_T (x/(T+1))^\alpha/\alpha$ for all $x > 0$. Combining this fact with (16) yields

$$\inf_{\nu\in\mathcal{M}} \inf_{\eta\in\mathcal{C}} \Lambda_{\eta,\nu} \geq \Big(\frac{1}{T+1} \Big)^\beta > 0. \tag{17}$$

Our next aim is to further simplify $\Lambda_{\eta,\nu}$. To this end, note that

$$
\begin{aligned}
&(D_t^\eta)^{1+\beta}(Z_t^\nu)^{-\beta}(S_t^0)^\beta \\
&= \mathcal{E}\Big(\int \big((1+\beta)\eta_{1s} + \beta\theta(Y_s)\big)\,dW_s^1 + \int \big((1+\beta)\eta_{2s} + \beta\nu_s\big)\,dW_s^2\Big)_t \qquad (18)\\
&\qquad \times \exp\Big(\int_0^t q(Y_s,\eta_s,\nu_s)\,ds\Big),
\end{aligned}
$$

where the function $q : \mathbb{R}\times\mathbb{R}^2\times\mathbb{R} \to [0,\infty[$ is given by

$$
q(y,\eta,\nu) = \frac{\beta(1+\beta)}{2}\big[(\eta_1+\theta(y))^2 + (\eta_2+\nu)^2\big] + \beta r(y).
$$

The Doleans-Dade exponential in (18) will be denoted by $\Delta_t^{\eta,\nu}$. If $\int_0^T \nu_t^2\,dt$ is bounded, then $\mathbb{E}[\,\Delta_T^{\eta,\nu}\,] = 1$. In general, however, we may have $\mathbb{E}[\,\Delta_T^{\eta,\nu}\,] < 1$ and this fact will create some technical difficulties in the sequel.

Our aim is to minimize $\Lambda_{\eta,\nu}$ over $\eta\in\mathcal{C}$ and $\nu\in\mathcal{M}_0$. To this end, for $t\geq 0$ and $\kappa\geq 0$, we introduce the measures

$$
\widetilde{\mu}_t(ds) := \kappa e^{\lambda(t-s)} \mathrm{I}_{[0,t]}(s)\,ds + \delta_t(ds)
$$

and, for $Y_0 = y$, the function

$$
\begin{aligned}
J(t,y,\eta,\nu) &:= \mathbb{E}\Big[\int (D_s^\eta)^{1+\beta}(Z_s^\nu)^{-\beta}(S_s^0)^\beta\,\widetilde{\mu}_t(ds)\Big] \\
&= \mathbb{E}\Big[\Delta_t^{\eta,\nu}\int \exp\Big(\int_0^s q(Y_r,\eta_r,\nu_r)\,dr\Big)\,\widetilde{\mu}_t(ds)\Big]
\end{aligned}
$$

so that by taking $\kappa := \gamma e^{-\lambda T}$ we get $J(T,Y_0,\eta,\nu) = n_T\Lambda_{\eta,\nu}$. To make the dependence of Y on its initial value explicit, we will sometimes also write Y^y for the solution of the SDE (3) with initial value $Y_0 = y$.

We will now use dynamic programming methods to solve the stochastic control problem with value function defined by

$$
V(t,y) := \inf_{\nu\in\mathcal{M}}\inf_{\eta\in\mathcal{C}} J(t,y,\eta,\nu).
$$

By taking $T := t$ and $\gamma := \kappa e^{\lambda t}$, the inequality (17) yields

$$
V(t,y) \geq n_t\Big(\frac{1}{t+1}\Big)^\beta > 0 \qquad \text{for all } t,y. \tag{19}
$$

For simplicity, we denote

$$
a(y) := \frac{1}{2}(\rho^2(y) + \varsigma^2(y)) \qquad \text{and} \qquad \widetilde{g}(y) := g(y) + \beta\rho(y)\theta(y).
$$

Theorem 3.1 *The function $V(t,y)$ is the unique bounded and strictly positive classical solution of the HJB equation*

$$
v_t = \kappa e^{\lambda t} + a v_{yy} + \widetilde{g} v_y + \inf_{\nu\in\mathbb{R}}\inf_{\eta\in\Gamma}\Big(\big[\rho(1+\beta)\eta_1 + \varsigma\big((1+\beta)\eta_2 + \beta\nu\big)\big]v_y + q(\cdot,\eta,\nu)v\Big) \tag{20}
$$

with initial condition

$$
v(0,y) = 1.
$$

The proof of this theorem will be prepared by several auxiliary lemmas. The first one deals with the possibility $\mathbb{E}[\,\Delta_T^{\eta,\nu}\,] < 1$. This happens when Z^ν is only a local martingale and not a true martingale. To deal with this situation, we will follow Föllmer [6, 7] and introduce the enlarged sample space $\bar\Omega := \Omega\times]0,\infty]$ endowed with the filtration

$$\bar{\mathcal{F}}_t := \sigma\big(A\times]s,\infty]\,|\,A\in\mathcal{F}_s,\ s\le t\big).$$

A finite $(\mathcal{F}_t)$-stopping time τ is lifted up to an $(\bar{\mathcal{F}}_t)$-stopping time $\bar\tau$ by setting $\bar\tau(\omega,s) := \tau(\omega)\mathrm{I}_{]\tau(\omega),\infty]}(s)$. Now let $\nu\in\mathcal{M}$ be given. Although we may have $\mathbb{E}[\,Z_T^\nu\,]<1$ it is possible to associate Z^ν with a *probability* measure $\bar P_\nu$ on $(\bar\Omega,\bar{\mathcal{F}}_\infty)$, where $\bar{\mathcal{F}}_\infty = \sigma(\bigcup_t\bar{\mathcal{F}}_t)$ as usual. This measure is called the *Föllmer measure* associated with the positive supermartingale Z^ν, and it is characterized by

$$\bar P_\nu[\,A\times]t,\infty]\,] = \mathbb{E}[\,Z^\nu_{t\wedge T}\mathrm{I}_A\,],\qquad 0\le t,\ A\in\mathcal{F}_t;$$

see [6, 7]. This identity carries over to the case in which the deterministic time t is replaced by a stopping time τ.

Lemma 3.2 *Suppose $\eta\in\mathcal{C}$ and $\nu\in\mathcal{M}$ are given, and (σ_n) is a localizing sequence for the local $\mathbb{P}$-martingale Z^ν. Then*

$$\mathbb{E}\big[\,(D^\eta_{t\wedge\sigma_n})^{1+\beta}(Z^\nu_{t\wedge\sigma_n})^{-\beta}(S^0_{t\wedge\sigma_n})^\beta\,\big]\nearrow\mathbb{E}\big[\,(D^\eta_t)^{1+\beta}(Z^\nu_t)^{-\beta}(S^0_t)^\beta\,\big].$$

In particular, the integrands converge in $L^1(\mathbb{P})$ if $\mathbb{E}[\,(D^\eta_t)^{1+\beta}(Z^\nu_t)^{-\beta}(S^0_t)^\beta\,]<\infty$.

Proof: Since $(S^0_{t\wedge\sigma_n})^\beta$ increases to the bounded random variable $(S^0_t)^\beta$, we may assume $r\equiv 0$ without loss of generality. Let Q be the probability measure in $\mathcal{Q}$ associated with η, and let us write $D := D^\eta$ and $Z := Z^\nu$.

First, we clearly have

$$\liminf_{n\uparrow\infty}\mathbb{E}\big[\,(D_{t\wedge\sigma_n})^{1+\beta}(Z_{t\wedge\sigma_n})^{-\beta}\,\big]\ge\mathbb{E}\big[\,(D_t)^{1+\beta}(Z_t)^{-\beta}\,\big]\tag{21}$$

due to Fatou's lemma.

Next, let $\bar P_\nu$ be the Föllmer measure associated with the positive supermartingale Z and let $\bar Q := Q\otimes\delta_\infty$ the extension of Q to $(\bar\Omega,\bar{\mathcal{F}}_\infty)$. Since Z is strictly positive, we obtain that for $t\le T$ and $A\in\mathcal{F}_t$

$$\bar Q[\,A\times]t,\infty]\,] = \mathbb{E}[\,D_t\mathrm{I}_A\,] = \mathbb{E}\Big[\,Z_t\frac{D_t}{Z_t}\mathrm{I}_A\,\Big] = \int\frac{D_t(\omega)}{Z_t(\omega)}\mathrm{I}_A(\omega)\mathrm{I}_{]t,\infty]}(s)\,\bar P_\nu(d\omega,ds).$$

Hence, $\overline{Q}\ll\bar P_\nu$ and the density process is given by

$$\frac{d\bar Q}{d\bar P_\nu}\Big|_{\bar{\mathcal{F}}_t}(\omega,s) = \frac{D_t(\omega)}{Z_t(\omega)}\mathrm{I}_{]t,\infty]}(s),\qquad t\le T.$$

Replacing t by a stopping time $\tau \leq T$ on the right, we thus obtain the density of $\bar{Q}$ with respect to $\bar{P}_\nu$ on $\bar{\mathcal{F}}_{\bar{\tau}}$, due to the optional stopping theorem. Hence, for two stopping times $\sigma \leq \tau \leq T$,

$$\begin{aligned}\mathbb{E}\big[\,(D_\tau)^{1+\beta}(Z_\tau)^{-\beta}\,\big] &= \int \Big(\frac{D_\tau(\omega)}{Z_\tau(\omega)}\Big)^\beta \mathrm{I}_{]\tau(\omega),\infty]}(s)\,\bar{Q}(d\omega, ds)\\ &= E_{\bar{P}_\nu}\Big[\,\Big(\frac{d\bar{Q}}{d\bar{P}_\nu}\Big|_{\bar{\mathcal{F}}_{\bar{\tau}}}\Big)^{1+\beta}\,\Big]\\ &\geq E_{\bar{P}_\nu}\Big[\,\Big(\frac{d\bar{Q}}{d\bar{P}_\nu}\Big|_{\bar{\mathcal{F}}_{\bar{\sigma}}}\Big)^{1+\beta}\,\Big]\\ &= \mathbb{E}\big[\,(D_\sigma)^{1+\beta}(Z_\sigma)^{-\beta}\,\big],\end{aligned}$$

where the inequality follows from Jensen's inequality for conditional expectations, and the last identity follows by reversing our previous steps. In particular, $\mathbb{E}[\,(D_{t\wedge\sigma_n})^{1+\beta}(Z_{t\wedge\sigma_n})^{-\beta}\,]$ is increasing in n and bounded above by $\mathbb{E}[\,(D_t)^{1+\beta}(Z_t)^{-\beta}\,]$. By combining this fact with (21), the result follows. □

The following lemma is a version of a standard verification result. Later on, it will first be applied with the choice $I := [-M, M]$, which corresponds to restricting the control space for ν in (20). The fact that I is compact will later on allow us to apply existence results for classical solutions v^I of the corresponding HJB equation.

We will say that a function $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ is of polynomial growth if there exist constants c and $p \geq 0$ such that $|v^I(t,y)| \leq c(1 + |y|^p)$ for all $y \in \mathbb{R}$ and $0 \leq t \leq T$.

Lemma 3.3 *Let I be a nonempty closed real interval, and suppose that the HJB equation*

$$v_t = \kappa e^{\lambda t} + a v_{yy} + \widetilde{g} v_y + \inf_{\nu \in I}\inf_{\eta\in\Gamma}\Big(\big[\rho(1+\beta)\eta_1 + \varsigma\big((1+\beta)\eta_2 + \beta\nu\big)\big]v_y + q(\cdot,\eta,\nu)v\Big) \quad (22)$$

admits a classical solution v^I of polynomial growth satisfying the initial condition

$$v^I(0,y) = 1. \quad (23)$$

In case I is non-compact, we assume in addition that v^I is bounded and strictly positive. Then we have $v^I(t,y) = V^I(t,y)$, where

$$V^I(t,y) := \inf_{\eta\in\mathcal{C}}\inf_{\nu\in\mathcal{M}^I} J(t,y,\eta,\nu)$$

for $\mathcal{M}^I$ denoting the set of all I-valued $\nu \in \mathcal{M}_0$. In particular, we have

$$v^I(t,y) \geq n_t\Big(\frac{1}{t+1}\Big)^\beta \qquad \textit{for } t \leq T \textit{ and } y \in \mathbb{R}. \quad (24)$$

Proof: Let us write $v = v^I$ throughout the proof. Let $\eta \in \mathcal{C}$ and $\nu \in \mathcal{M}^I$ be controls such that such that $J(u, y, \eta, \nu) < \infty$ and define

$$dM_s := \rho(Y_s)\, dW_s^1 + \varsigma(Y_s)\, dW_s^2 - \rho(Y_s)\big((1+\beta)\eta_{1s} + \beta\theta(Y_s)\big)\, ds - \varsigma(Y_s)\big((1+\beta)\eta_{2s} + \beta\nu_s\big)\, ds.$$

Then the SDE for Y can be rewritten as

$$dY_s = dM_s + \Big\{\widetilde{g}(Y_s) + \rho(Y_s)(1+\beta)\eta_{1s} + \varsigma(Y_s)\big((1+\beta)\eta_{2s} + \beta\nu_s\big)\Big\}\, ds.$$

For any $\widetilde{\nu} \in I$ and $\widetilde{\eta} \in \Gamma$ we define a differential operator $\mathcal{A}^{\widetilde{\eta},\widetilde{\nu}}$ by

$$\mathcal{A}^{\widetilde{\eta},\widetilde{\nu}} = -\partial_t + a\partial_{yy} + \Big(\widetilde{g} + \rho(1+\beta)\widetilde{\eta}_1 + \varsigma\big((1+\beta)\widetilde{\eta}_2 + \beta\widetilde{\nu}\big)\Big)\partial_y.$$

Then, by Itô's formula and (22),

$$\begin{aligned} &d\Big[e^{\int_0^t q(Y_s,\eta_s,\nu_s)\, ds} v(u-t, Y_t)\Big] \\ &= e^{\int_0^t q(Y_s,\eta_s,\nu_s)\, ds}\Big[v_y(u-t, Y_t)\, dM_t + \Big(\mathcal{A}^{\eta_t,\nu_t} v(u-t, Y_t) + q(Y_t, \eta_t, \nu_t) v(u-t, Y_t)\Big)\, dt\Big] \\ &\geq e^{\int_0^t q(Y_s,\eta_s,\nu_s)\, ds}\Big[v_y(u-t, Y_t)\, dM_t - \kappa e^{\lambda(u-t)}\, dt\Big]. \end{aligned} \tag{25}$$

Next let

$$\sigma_n := \inf\Big\{\, t \geq 0 \,\Big|\, |v_y((u-t)^+, Y_t)| \geq n \text{ or } \int_0^t \nu_s^2\, ds \geq n \Big\}.$$

Then (σ_n) is a localizing sequence for the local $\mathbb{P}$-martingale Z^ν. Defining a probability measure P^n by $dP^n = \Delta^{\eta,\nu}_{u\wedge\sigma_n}\, d\mathbb{P}$, it follows from Girsanov's theorem that $(M_t^{\sigma_n})_{0\leq t\leq u}$ is a P^n-martingale. By taking expectations with respect to P^n, we hence get

$$v(u, Y_0) \leq E^n\Big[\, e^{\int_0^{u\wedge\sigma_n} q(Y_s,\eta_s,\nu_s)\, ds}\, v(u - u\wedge\sigma_n, Y_{u\wedge\sigma_n}) + \int_0^{u\wedge\sigma_n} \kappa e^{\lambda(u-t)} e^{\int_0^t q(Y_s,\eta_s,\nu_s)\, ds}\, dt\,\Big]. \tag{26}$$

We will first look at the second term on the right:

$$\begin{aligned} &E^n\Big[\int_0^{u\wedge\sigma_n} \kappa e^{\lambda(u-t)} e^{\int_0^t q(Y_s,\eta_s,\nu_s)\, ds}\, dt\Big] \\ &= \int_0^u \kappa e^{\lambda(u-t)} \mathbb{E}\Big[\,\Delta^{\eta,\nu}_{t\wedge\sigma_n} e^{\int_0^t q(Y_s,\eta_s,\nu_s)\, ds}\, \mathrm{I}_{\{t\leq\sigma_n\}}\,\Big]\, dt \\ &= \int_0^u \kappa e^{\lambda(u-t)} \mathbb{E}\Big[\,(D^\eta_{t\wedge\sigma_n})^{1+\beta} (Z^\nu_{t\wedge\sigma_n})^{-\beta} (S^0_{t\wedge\sigma_n})^\beta\, \mathrm{I}_{\{t\leq\sigma_n\}}\,\Big]\, dt, \end{aligned}$$

and an application of Lemma 3.2, together with monotone convergence and our assumption $J(u, y, \eta, \nu) < \infty$, implies that the latter expression converges to

$$\int_0^u \kappa e^{\lambda(u-t)} E_Q\Big[\,(D_t^\eta)^\beta (Z_t^\nu)^{-\beta} (S_t^0)^\beta\,\Big]\, dt.$$

The first expectation in (26) is equal to

$$E_Q\Big[\,(D^\eta_{u\wedge\sigma_n})^\beta (Z^\nu_{u\wedge\sigma_n})^{-\beta} (S^0_{u\wedge\sigma_n})^\beta v(u - u\wedge\sigma_n, Y_{u\wedge\sigma_n})\,\Big]. \tag{27}$$

We will argue below that the integrands in (27) are uniformly integrable with respect to Q. Due to the initial condition (23) and the continuity of v, we will thus get

$$v(u, Y_0) \le E_Q\Big[\int (D_t^\eta)^\beta (Z_t^\nu)^{-\beta} (S_t^0)^\beta \, \widetilde{\mu}_u(dt) \Big] = J(u, y, \eta, \nu) \tag{28}$$

and in turn $v \le V^I$.

Let us now show that the integrands in (27) are uniformly integrable. For unbounded I, this follows from the boundedness of v, Lemma 3.2, and our assumption $J(u, y, \eta, \nu) < \infty$. For bounded I, one easily shows that the integrands have uniformly bounded $L^2(Q)$-norms. Indeed, we have

$$\begin{aligned} &E_Q\Big[(D^\eta_{t\wedge\sigma_n})^{2\beta} (Z^\nu_{t\wedge\sigma_n})^{-2\beta} (S^0_{t\wedge\sigma_n})^{2\beta} v(u - u\wedge\sigma_n, Y_{u\wedge\sigma_n})^2 \Big] \\ &\le \; E_Q\Big[(D^\eta_{t\wedge\sigma_n})^{4\beta} (Z^\nu_{t\wedge\sigma_n})^{-4\beta} (S^0_{t\wedge\sigma_n})^{4\beta} \Big]^{1/2} E_Q\Big[v(u - u\wedge\sigma_n, Y_{u\wedge\sigma_n})^4 \Big]^{1/2}. \end{aligned}$$

The uniform boundedness of the first term on the right now follows by an application of Lemma 3.2 for $\beta' := 4\beta$. The second term can be bounded in the form $C(1 + E_Q[\,|Y_{u\wedge\sigma_n}|^{4p}\,])$, due to the polynomial growth condition of v. It is well known and easy to show that, under the original measure $\mathbb{P}$, the random variable $\sup_{t\le T}|Y_t|$ has moments of all orders. Since the process η is bounded, the same is true under Q, and the desired uniform integrability follows.

In order to prove the reverse inequality $v \ge V^I$, let us first consider the case of a compact interval I. Due to compactness, we then may find Markov controls

$$(\eta^*, \nu^*) \in \underset{\nu\in I, \eta\in\Gamma}{\arg\min} \Big\{ \big[\rho(1+\beta)\eta_1 + \varsigma\big((1+\beta)\eta_2 + \beta\nu\big)\big] v_y + q(\cdot, \eta, \nu) v \Big\},$$

which by a measurable selection argument can be chosen as measurable functions $\eta^*(t, y)$, $\nu^*(t, y)$ of t and y. Using the controls $\nu^*_s := \nu^*(u-s, Y_s) \in \mathcal{M}^I$, $\eta^*_s := \eta^*(u-s, Y_s) \in \mathcal{C}$, we get an equality in (25) and hence in (26) and (28). Thus, $v(t, y) \ge J(t, y, \eta^*, \nu^*) \ge V^I(t, y)$. In particular, (24) follows from (19).

If I is unbounded, we note first that the supremum of the nonlinear term in (22) with respect to all $\nu \in \mathbb{R}$ is attained in

$$\widehat{\nu} = -\eta_2 - \frac{\varsigma}{1+\beta} \cdot \frac{v_y}{v}, \tag{29}$$

which is always well-defined, due to our hypothesis of strict positivity of v. Hence, the supremum with respect to $\nu \in I$ is also attained, and we can define processes $\nu^*_s := \nu^*(u - s, Y_s)$ and $\eta^*_s := \eta^*(u - s, Y_s)$ as above, for which we get an equality in (25). We clearly have $\eta^* \in \mathcal{C}$ and that ν^* is I-valued. In addition, for any (t, y), the function $\nu^*(t, y)$ is either of the form (29) with η_2 replaced by $\eta_2^*(t, y)$ or takes its value in the boundary of I, and so the boundedness of η^*_2, the continuity of v_y and v, and the strict positivity of v imply that $\int_0^T \nu^*(T-t, Y_t)^2\,dt < \infty$ along any continuous sample path of Y. This yields an equality in (28). $\square$

According to [5, Theorem IV.4.3 and Remark IV.4.1], the equation (22)–(23) admits a unique classical solution v^I of polynomial growth as soon as I is compact. By the preceding lemma, this solution is equal to the value function V^I. Our goal is to show that the unconstrained value function V can be obtained as an appropriate limit of the functions $v^I = V^I$ when $I \uparrow \mathbb{R}$. To this end, we will prove some a priori estimates, which hold uniformly with respect to I.

Lemma 3.4 *Suppose I is a compact real interval containing 0. Then,*

$$0 \le v^I_t(t,y) \le C_1 v^I(t,y),$$

where

$$C_1 := \inf_{x\in\Gamma} \big(\|q(\cdot,x,0)\|_\infty + e^\lambda(\kappa+\lambda)\big)e^{\|q(\cdot,x,0)\|_\infty}.$$

In particular, v^I is uniformly bounded on $[0,T]\times\mathbb{R}$:

$$1 \le v^I(t,y) \le e^{C_1 T}.$$

Proof: We will use the representation of v^I as the value function V^I. Let us take $\delta \in]0,1]$ such that $0 \le t+\delta \le T$. Since I is compact, $\Delta^{\eta,\nu}$ is a $\mathbb{P}$-martingale for all $\eta \in \mathcal{C}$ and $\nu \in \mathcal{M}^I$. Hence, in proving the lower bound we may argue that

$$V^I(t+\delta,y) - V^I(t,y) \ge \inf_{\nu\in\mathcal{M}^I,\eta\in\mathcal{C}} \big[J(t+\delta,y,\eta,\nu) - J(t,y,\eta,\nu)\big]$$
$$= \inf_{\nu\in\mathcal{M}^I,\eta\in\mathcal{C}} \mathbb{E}\Big[\Delta^{\eta,\nu}_{(t+\delta)}\Big(\int e^{\int_0^s q(Y_u,\eta_u,\nu_u)\,du}\,\widetilde{\mu}_{t+\delta}(ds) - \int e^{\int_0^s q(Y_u,\eta_u,\nu_u)\,du}\,\widetilde{\mu}_t(ds)\Big)\Big],$$

and one easily sees that the difference of the two integrals is nonnegative, due to our assumption $r \ge 0$.

To prove the upper bound, take $\varepsilon > 0$, $x \in \Gamma$, and processes $\widetilde{\nu} \in \mathcal{M}^I$ and $\widetilde{\eta} \in \mathcal{C}$ such that $V^I(t,y) + \varepsilon\delta \ge J(t,y,\widetilde{\eta},\widetilde{\nu})$ and, for $s \in [t,t+\delta]$, $\widetilde{\nu}_s = 0$ and $\widetilde{\eta}_s = x$. It follows from Lemma 3.2 that

$$\begin{aligned}
&V^I(t+\delta,y) - V^I(t,y) - \varepsilon\delta \\
&\le J(t+\delta,y,\widetilde{\eta},\widetilde{\nu}) - J(t,y,\widetilde{\eta},\widetilde{\nu}) \\
&= \mathbb{E}\Big[\Delta^{\widetilde{\eta},\widetilde{\nu}}_{t+\delta}\Big\{e^{\int_0^t q(Y_s,\widetilde{\eta}_s,\widetilde{\nu}_s)\,ds}\Big(e^{\int_t^{t+\delta} q(Y_s,x,0)\,ds} - 1\Big) \\
&\qquad + \kappa\int_0^t e^{\lambda(t-s)}e^{\int_0^s q(Y_u,\widetilde{\eta}_u,\widetilde{\nu}_u)\,du}(e^{\lambda\delta}-1)\,ds \\
&\qquad + \kappa\int_t^{t+\delta} e^{\lambda(t+\delta-s)}e^{\int_0^t q(Y_u,\widetilde{\eta}_u,\widetilde{\nu}_u)\,du}e^{\int_t^s q(Y_u,x,0)\,du}\,ds\Big\}\Big] \\
&\le \delta J(t,y,\widetilde{\eta},\widetilde{\nu})\big(\|q(\cdot,x,0)\|_\infty + e^\lambda(\kappa+\lambda)\big)e^{\|q(\cdot,x,0)\|_\infty},
\end{aligned}$$

which gives the upper bound. □

Lemma 3.5 *Suppose that I is a compact nonempty real interval containing zero, and v^I is the classical solution of polynomial growth to* (22)*–*(23)*. Then there exists a constant C_2, depending only on α, κ, λ, Γ, and the coefficients in* (1)*–*(3)*, such that $|v^I_y| \le C_2(1+|y|)$ and $|v^I_{yy}| \le C_2(1+|y|^2)$.*

Proof: Let $w := \log v^I = \log V^I \ge 0$. We have $|v^I_y| = v^I|w_y|$ and $|v^I_{yy}| \le v^I(|w_{yy}| + w_y^2)$. Since $v^I \le e^{C_1 T}$ by Lemma 3.4, it is sufficient to obtain analogous estimates on $|w_y|$ and $|w_{yy}|$ from above. The function w satisfies the equation

$$\begin{aligned} w_t = \kappa e^{\lambda t} e^{-w} &+ a(w_{yy} + w_y^2) + (g + \beta\rho\theta) w_y \\ &+ \inf_{\nu\in I}\inf_{\eta\in\Gamma}\Big(\big[\rho(1+\beta)\eta_1 + \varsigma\big((1+\beta)\eta_2 + \beta\nu\big)\big]w_y + q(\cdot,\eta,\nu)\Big) \end{aligned} \tag{30}$$

with initial condition

$$w(0,\cdot) \equiv 0.$$

Moreover, we have

$$0 \le w_t \le C_1, \tag{31}$$

due to Lemma 3.4.

Next, the boundedness of w implies that, for fixed t, the function $y \mapsto |w_y(t,y)|$ cannot tend towards its supremum as $y \uparrow \infty$ or $y \downarrow -\infty$. Hence, it is enough to estimate the function $w_y(t,y)$ in its critical points. In these points, we have

$$w_t = \kappa e^{\lambda t} e^{-w} + a w_y^2 + \widetilde{g} w_y + \phi^I(w_y), \tag{32}$$

where ϕ^I denotes the infimum in (30), considered as a function of w_y (and implicitly also of y). When taking the infimum over *all* $\nu \in \mathbb{R}$ one finds that

$$0 \ge \phi^I(y,p) \ge -\frac{1}{2}\alpha\varsigma^2(y)p^2 + \psi(y,p), \qquad p \in \mathbb{R}, \tag{33}$$

where

$$\psi(y,p) := \inf_{\eta\in\Gamma}\Big(\big[\rho(y)(1+\beta)\eta_1 + \varsigma(y)\eta_2\big]p + \frac{\beta(1+\beta)}{2}(\eta_1 + \theta(y))^2\Big).$$

By using the upper bound in (31) and the lower bound in (33), we obtain

$$C_1 \ge \frac{1}{2}(\rho^2 + (1-\alpha)\varsigma^2)w_y^2 + \widetilde{g} w_y + \psi(w_y).$$

Next, due to the compactness of Γ, we have $|\psi(y,p)| \le c_1(1+|p|)$ for a constant c_1 depending on Γ, α, $\|\rho\|_\infty$, $\|\varsigma\|_\infty$, and $\|\theta\|_\infty$. Using the fact that $\widetilde{g}(y)$ grows at most linearly in y, we thus get

$$C_1 \ge \frac{1}{2}(1-\alpha)\sigma_1^2 w_y^2(t,y) - c_2\big(1 + |w_y(t,y)|(1+|y|)\big),$$

where σ_1 is as in (4) and c_2 is an appropriate constant depending on c_1, g, α, $\|\rho\|_\infty$, and $\|\theta\|_\infty$. Hence,

$$\sqrt{c_3 + c_4^2(1+|y|)^2} \ge \big|w_y(t,y) - c_4(1+|y|)\big|,$$

where c_3 and c_4 depend on C_1, c_2, α, and σ_1, and from here the estimate on $|w_y|$ follows. Also the one on $|w_{yy}|$ is now straightforward. □

Proof of Theorem 3.1: We first restrict the control space for ν to some bounded interval $I := [-M, M]$. As mentioned above, this guarantees the existence of a classical solution v^I of the constrained HJB equation (22)–(23) such that v^I has at most polynomial growth. By Lemma 3.3, this solution is unique and corresponds to the value function V^I. Moreover, it is bounded and ≥ 1 according to Lemma 3.4. As observed in (29), the supremum with respect to $\nu \in I$ in (22) is achieved at

$$\widehat{\nu} = -\eta_2 - \frac{\varsigma}{1+\beta} \cdot \frac{V_y^I}{V^I}, \tag{34}$$

when this expression belongs to the set I. Otherwise it will be achieved in the extremes of this set. By Lemma 3.5, $\widehat{\nu}$ will be given by (34) as soon as

$$M \geq M(y) := \max_{\eta\in\Gamma} |\eta_2| + \frac{\|\varsigma\|_\infty C_2}{1+\beta}(1+|y|).$$

Thus, denoting $I_n := [-M(n), M(n)]$ and $v^n := v^{I_n}$, we conclude that v^n locally satisfies the unconstrained HJB equation, i.e.,

$$v_t^n = \kappa e^{\lambda t} + a v_{yy}^n + \widetilde{g} v_y^n + v^n \phi(v_y^n / v^n), \qquad \text{for } |y| \leq n,$$

with

$$\phi(p) := \inf_{\nu\in\mathbb{R}} \inf_{\eta\in\Gamma} \Big(\big[\rho(1+\beta)\eta_1 + \varsigma\big((1+\beta)\eta_2 + \beta\nu\big)\big]p + q(\cdot, \eta, \nu) \Big).$$

It follows from the definition of the value functions that the functions $v^n = V^{I_n}$ pointwise decrease to a function v satisfying $1 \leq v \leq e^{C_1 T}$. Since the gradients v_y^n and time derivatives v_t^n are locally uniformly bounded by Lemmas 3.5 and 3.4, it follows from the Arzela-Ascoli theorem that convergence holds even locally uniformly in $C([0,T]\times\mathbb{R})$. Moreover, by Lemma 3.5 also v_{yy}^n is locally uniformly bounded. For each t, another application of the Arzela-Ascoli theorem thus yields the existence of a subsequence $(v^{n_k}(t,\cdot))$ such that $(v_y^{n_k}(t,\cdot))$ converges locally uniformly in $C(\mathbb{R})$ to $v_y(t,\cdot)$, hence $v \in C^{0,1}([0,T]\times\mathbb{R})$. Furthermore, the locally uniform bounds on v_t^n, v_y^n, and v_{yy}^n imply that v is locally Lipschitz continuous on $[0,T]\times\mathbb{R}$ with $|v_t| \leq C_1 v$ a.e. on $[0,T]\times\mathbb{R}$ and $|v_y(t,y)| \leq C_2(1+|y|)$ for all $t \leq T$ and $y \in \mathbb{R}$. Moreover,

$$|v_y(t,y) - v_y(t,y')| \leq C_2(1+K^2)|y-y'| \qquad \text{for } y, y' \in [-K, K].$$

Next, let $f_n(t,y) := \kappa e^{\lambda t} + v^n(t,y)\phi^{I_n}(v_y^n(t,y)/v^n(t,y))$, so that the equation for v^n can be written as $v_t^n = a v_{yy}^n + \widetilde{g} v_y^n + f_n$. Since v^n belongs to $C^{1,2}([0,T]\times\mathbb{R})$ and f_n has at most linear growth in y, we obtain the stochastic representation

$$v^n(t,y) = 1 + \mathbb{E}\Big[\int_0^t f_n(s, \widetilde{Y}_s^y)\, ds\Big],$$

where $\widetilde{Y}$ solves (3) with g replaced by $\widetilde{g}$. In fact, Lemma 3.5 even yields $|f_n(t,y)| \leq C_3(1+|y|^2)$ uniformly in n, $t \leq T$, and $y \in \mathbb{R}$ for some constant C_3. Hence, using the convergence of v^n and v_y^n and passing to the limit with dominated convergence, combined with the fact that $\sup_{s\le t}|\widetilde{Y}_s^y|$ has moments of all orders, yields

$$v(t,y) = 1 + \mathbb{E}\Big[\int_0^t f(s,\widetilde{Y}_s^y)\,ds\Big],$$

where $f(t,y) := \kappa e^{\lambda t} + v(t,y)\phi(v_y(t,y)/v(t,y))$. If we can show that $(t,y)\mapsto f(t,y)$ is continuous, then, since f satisfies a local Lipschitz condition in y uniformly in $t\le T$, Theorem 12 on p. 25 of [11] will imply that v is a bounded $C^{1,2}$-solution of the linear parabolic equation $v_t = av_{yy} + \widetilde{g}v_y + f$ and in turn of (20). Moreover, Lemma 3.3 will yield the identification $v = V$.

To prove the continuity of f, let us fix a flow of $(\widetilde{Y}_t^y)_{y\in\mathbb{R},\,t\ge 0}$ so that we have

$$\frac{\partial \widetilde{Y}_t^y}{\partial y} = e^{\int_0^t g'(\widetilde{Y}_s^y)\,ds}\cdot\mathcal{E}\Big(\int_0 \rho'(\widetilde{Y}_s^y)\,dW_s^1 + \int_0 \varsigma'(\widetilde{Y}_s^y)\,dW_s^2\Big)_t.$$

The stochastic exponential on the right is the density process with respect to $\mathbb{P}$ of a probability measure $\widetilde{\mathbb{P}}$ under which $\widetilde{Y}$ solves the SDE

$$d\widetilde{Y}_t^y = \rho(\widetilde{Y}_t^y)\,d\widetilde{W}_t^1 + \varsigma(\widetilde{Y}_t^y)\,d\widetilde{W}_t^2 + h(\widetilde{Y}_t^y)\,dt$$

for two independent $\widetilde{\mathbb{P}}$-Brownian motions $\widetilde{W}^i$, $i=1,2$, and with $h = g + \rho\rho' + \varsigma\varsigma'$. Note that $y\mapsto f(s,y)$ is locally Lipschitz continuous on $[-K,K]$ with a Lipschitz constant that is uniform in $t\in[0,T]$ and growths at most as a constant times K^4. Hence, dominated convergence implies that

$$v_y(t,y) = \mathbb{E}\Big[\int_0^t f_y(s,\widetilde{Y}_s^y)\frac{\partial\widetilde{Y}_s^y}{\partial y}\,ds\Big] = \int_0^t \widetilde{\mathbb{E}}\big[\,f_y(s,\widetilde{Y}_s^y)e^{\int_0^s g'(\widetilde{Y}_u^y)\,du}\,\big]\,ds.$$

The latter expression is Lipschitz continuous in t, locally uniformly in y. Together with the already established local Lipschitz continuity of $y\mapsto v_y(t,y)$, which holds uniformly in $t\in[0,T]$, we obtain the continuity of $(t,y)\mapsto v_y(t,y)$, which in turn yields the continuity of $f = \kappa + v\phi(v_y/v)$. □

Proof of Theorem 2.2: First, one easily checks that by taking the minimum over $\nu\in\mathbb{R}$ the two equations (9) and (20) become equivalent when taking $\kappa := \gamma e^{-\lambda T}$. So let v be the solution of (20).

To compute the optimal strategy $(\widehat{c},\widehat{\pi})$, recall from (14) and (15) that the optimal consumption process and the optimal wealth process $X_T^{x,\widehat{c}\widehat{\pi}}$ are given by

$$\widehat{c}_t = \frac{1}{n_T}\gamma e^{-\lambda t} I\Big(\frac{\widehat{z}Z_t^{\widehat{\nu}}}{D_t^{\widehat{\eta}}S_t^0}\Big) \qquad\text{and}\qquad X_T^{x,\widehat{c},\widehat{\pi}} = \frac{1}{n_T} I\Big(\frac{\widehat{z}Z_T^{\widehat{\nu}}}{D_T^{\widehat{\eta}}S_T^0}\Big),$$

where $I(y) = -\widetilde{U}'(y) = y^{-\beta-1}$, $\widehat{\eta}_t = \eta^*(T-t,Y_t)$ and $\widehat{\nu}_t = \nu^*(T-t,Y_t)$ are optimal Markovian controls for (20) and

$$\widehat{z} = \Big(\frac{\Lambda_{\widehat{\eta},\widehat{\nu}}}{x}\Big)^{1/(1+\beta)} = \Big(\frac{v(T,Y_0)}{n_T x}\Big)^{1/(1+\beta)}.$$

Let us show next that $Z^{\widehat{\nu}}$ is a true $\mathbb{P}$-martingale. First, it follows from (29) and our bounds on the solution v that $|\widehat{\nu}_t| \le C(1+|Y_t|)$ for some constant C. Since by [21, Theorem 4.7] there exists $\delta > 0$ such that $\sup_{0\le t\le T}\mathbb{E}\big[\exp(\delta|Y_t|)\big] < \infty$, we obtain $\sup_{0\le t\le T}\mathbb{E}\big[\exp(\varepsilon|\widehat{\nu}_t|)\big] < \infty$ for $\varepsilon = \delta/C$. According to [21], p. 220, the martingale property of $Z^{\widehat{\nu}}$ follows.

Next, by arguing as in the proof of [25, Theorem 2.5] and using the duality relations as stated in [28, Theorem 2.5], one shows that

$$M_t := \Big(\frac{X_t^{x,\widehat{c},\widehat{\pi}}}{S_t^0} + \int_0^t \frac{\widehat{c}_s}{S_s^0}\,ds\Big)Z_t^{\widehat{\nu}}$$

is a true $\mathbb{P}$-martingale. Since M and $Z^{\widehat{\nu}}$ are martingales, equation (6) yields that

$$dM_t - \frac{M_t}{Z_t^{\widehat{\nu}}}\,dZ_t^{\widehat{\nu}} = \Big[M_t - Z_t^{\widehat{\nu}}\int_0^t \frac{\widehat{c}_s}{S_s^0}\,ds\Big]\widehat{\pi}_t\sigma(Y_t)\,dW_t^1, \tag{35}$$

where the computation can be simplified by noting that all finite-variation terms must cancel out, due to the martingale property. On the other hand, by the martingale property of $Z^{\widehat{\nu}}$,

$$M_t = \mathbb{E}[\,M_T\,|\,\mathcal{F}_t\,] = Z_t^{\widehat{\nu}}\int_0^t \frac{\widehat{c}_s}{S_s^0}\,ds + \frac{\widehat{z}^{-\beta-1}}{n_T}(Z_t^{\widehat{\nu}})^{-\beta}(D_t^{\widehat{\eta}})^{1+\beta}(S_t^0)^{\beta}\cdot\mathcal{E}_t,$$

where

$$\mathcal{E}_t \;=\; \mathbb{E}\Big[\int_t^T\Big(\frac{Z_s^{\widehat{\nu}}}{Z_t^{\widehat{\nu}}}\Big)^{-\beta}\Big(\frac{D_s^{\widehat{\eta}}}{D_t^{\widehat{\eta}}}\Big)^{1+\beta}\Big(\frac{S_s^0}{S_t^0}\Big)^{\beta}\,\widetilde{\mu}_T(ds)\,\Big|\,\mathcal{F}_t\Big].$$

Using the Markov property of Y and introducing the controls $\widehat{\eta}_s^{(t)} := \eta^*(T-t-s,Y_s)$ and $\widehat{\nu}_s^{(t)} := \nu^*(T-t-s,Y_s)$, we obtain

$$\mathcal{E}_t = J(T-t,Y_t,\widehat{\eta}^{(t)},\widehat{\nu}^{(t)}) = v(T-t,Y_t).$$

Moreover, we have $\widehat{z}^{-\beta-1} = xn_T/v(T,Y_0)$, and thus get

$$M_t = Z_t^{\widehat{\nu}}\int_0^t \frac{\widehat{c}_s}{S_s^0}\,ds + x(Z_t^{\widehat{\nu}})^{-\beta}(D_t^{\widehat{\eta}})^{1+\beta}(S_t^0)^{\beta}\cdot\frac{v(T-t,Y_t)}{v(T,Y_0)}. \tag{36}$$

This gives

$$X_t^{x,\widehat{c},\widehat{\pi}} = x\Big(\frac{Z_t^{\widehat{\nu}}}{D_t^{\widehat{\eta}}S_t^0}\Big)^{-1-\beta}\cdot\frac{v(T-t,Y_t)}{v(T,Y_0)} = \widehat{c}_t\frac{e^{\lambda t}}{\gamma}v(T-t,Y_t),$$

and this formula yields our claim for the form of $\widehat{c}_t$.

To prove the formula for $\widehat{\pi}$, we take differentials in (36) and get

$$\begin{aligned}
dM_t - \frac{M_t}{Z_t^{\widehat{\nu}}}\,dZ_t^{\widehat{\nu}} &= \Big[M_t - Z_t^{\widehat{\nu}}\int_0^t \frac{\widehat{c}_s}{S_s^0}\,ds\Big]\Big[(1+\beta)\Big((\theta(Y_t)+\widehat{\eta}_{1t})\,dW_t^1 + (\widehat{\nu}_t+\widehat{\eta}_{2t})\,dW_t^2\Big)\\
&\qquad\qquad + \frac{v_y(T-t,Y_t)}{v(T-t,Y_t)}\big(\rho(Y_t)\,dW_t^1 + \varsigma(Y_t)\,dW_t^2\big)\Big]\\
&= \Big[M_t - Z_t^{\widehat{\nu}}\int_0^t \frac{\widehat{c}_s}{S_s^0}\,ds\Big]\Big[(1+\beta)(\theta(Y_t)+\widehat{\eta}_{1t}) + \rho(Y_t)\frac{v_y(T-t,Y_t)}{v(T-t,Y_t)}\Big]\,dW_t^1
\end{aligned}$$

where the martingale property again significantly simplifies the computation and the second identity uses (34). Comparing this identity with (35) yields our formula for $\widehat{\pi}$ and completes the proof of Theorem 2.2. □

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