Optimal investments for risk- and ambiguity-averse preferences: a duality approach

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Abstract: Ambiguity, also called Knightian or model uncertainty, is a key feature in financial modeling. A recent paper by Maccheroni *et al.* (2004) characterizes investor preferences under aversion against both risk and ambiguity. Their result shows that these preferences can be numerically represented in terms of convex risk measures. In this paper we study the corresponding problem of optimal investment over a given time horizon, using a duality approach and building upon the results by Kramkov and Schachermayer (1999, 2001).

Key words: Model uncertainty, ambiguity, convex risk measures, optimal investments, duality theory

1 Introduction

In the vast majority of the literature on optimal investments in financial markets it is assumed that decisions are based on a classical expected utility criterion in the sense of John von Neumann and Oscar Morgenstern. Underlying this concept is the assumption that expected utility is computed in terms of a probability measure that accurately models future stock price evolutions. In reality, however, the choice of this probability measure is itself subject to *model uncertainty*, often also called *ambiguity* or *Knightian uncertainty*. Economists have long been aware of this fact, and in the late 1980's Gilboa and Schmeidler [36, 20] formulated axioms on investor preferences that should account for aversion against both risk and ambiguity. They showed that these preferences can be numerically represented by a 'coherent' robust utility functional of the form

$$X \longmapsto \inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,], \tag{1}$$

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where U is a utility function and $\mathcal{Q}$ is a class of probability measures; see also [18, Section 2.5] for a survey. The elements of $\mathcal{Q}$ can be interpreted as prior models, which possibly describe the probabilities of future scenarios. Taking the infimum of all expected utilities for these prior models thus corresponds to a worst-case approach. Systematic analyses of optimal investment decisions under this type of preferences were given, among others, by Talay and Zheng [38], Quenez [30], the author [32, 33], Burgert and Rüschendorf [5], Wu and the author [35], Föllmer and Gundel [15], Müller [29], and Hernández-Hernández and the author [22].

One might object that robust utility functionals of the form (1) leave no room for discriminating models in $\mathcal{Q}$ according to their plausibility. If, for instance, the class of prior models arises as a confidence set in statistical estimation, then the original estimate might have a higher plausibility, and thus should receive a higher weight, than a model at the boundary of the confidence set. Or one might wish to include the results of certain stress test models when their outcomes differ significantly from the ones of plausible priors; see, e.g., Carr et al. [6] and [18, Section 4.8]. These objections to robust utility functionals of the form (1) correspond to objections that can be raised on an axiomatic level against the axiom of 'certainty independence' introduced in [20]. By weakening this axiom, Maccheroni *et al.* [27] recently obtained a numerical representation of the form

$$X \longmapsto \inf_{Q} \big(E_Q[\, U(X) \,] + \gamma(Q) \big), \tag{2}$$

where the function γ assigns a penalization weight $\gamma(Q)$ to each possible probabilistic model Q. This class of robust utility functionals clearly extends the class (1) and leaves room for a discrimination among possible prior models. The move from (1) to (2) is similar to the generalization of coherent by convex risk measures [16, 17, 18, 19].

Our goal in this paper is to study the problem of constructing dynamic investment strategies whose terminal wealth maximizes a functional (2) for a given initial investment. More precisely, we will build on the results by Kramkov and Schachermayer [25, 26] and develop the duality theory for the maximization of the robust utility in a very general setting and under rather weak assumptions. Our main results are a minimax identity stating that the maximization over strategies and the minimization over measures can be interchanged, an analysis of the duality relations between the primal and the dual problems, and an existence and uniqueness result for optimal strategies based on a characterization of the optimal terminal wealth. The duality theory for 'coherent' robust utility functionals of the form (1) was first developed in [30] and later extended in [35]. Our results given here are stronger than the ones in [35] even when restricted to the 'coherent' case. In particular, we discuss the structure of the set of all solutions to the dual (and hence the primal) problem and prove the existence of a unique solution that satisfies a natural property of maximality. This discussion requires us to work with a setup of the dual problem, which is somewhat different from the one introduced in [35]. We also discuss in detail what happens if even the maximal solution does not have full support.

In many situations, the dual problem is simpler than the primal one, and so it can be advantageous to apply some kind of control approach to the dual rather than to the primal problem. This is already true for the maximization of classical von Neumann-Morgenstern utility, but in robust utility maximization there is the additional advantage that the dual problem simply involves the minimization of a convex functional while the primal problem requires to find a saddlepoint of a functional, which is concave in one argument and convex in the other. In the case of 'coherent' robust utility maximization, [30] combines duality techniques with a control approach based on backward stochastic differential equations, while in [22] a Hamilton-Jacobi-Bellman partial differential equation (PDE) is derived for the dual problem. Based on the duality results given in (an earlier version of) this paper, Hernández-Hernández and the author [23] recently obtained an explicit PDE characterization of the optimal strategy in an incomplete diffusion market model where the robust utility functional is defined in terms of a logarithmic utility function and a rather general dynamically consistent penalty function $\gamma(\cdot)$. This penalty function is described in Example 3.4.

Another feature of the duality method is that it works in a much more general setting than control techniques. The latter approach requires in particular the dynamic consistency of the underlying convex risk measure

$$\rho(Y) := \sup_Q \big(E_Q[\,-Y\,] - \gamma(Q)\big)$$

in the sense described, e.g., by Epstein and Schneider [14] for coherent risk measures and by Cheridito *et al.* [7] for the general case. This requirement of dynamic consistency rules out many examples, for which the dual approach makes perfect sense; see the discussion in Section 3 and in particular Remark 3.5. See also [32] for the explicit computation of optimal strategies in 'coherent' examples, which are not dynamically consistent.

In Section 2 we formulate our hypotheses and state our main results. As in standard expected utility maximization, we observe that the duality for the value functions of the robust problem holds under rather mild conditions, while a stronger condition is necessary to guarantee the existence of optimal strategies. In Section 3 we present possible choices for penalty functions $\gamma(\cdot)$. We also give examples showing that the value function of the robust problem may not be continuously differentiable. Equivalently, the dual value function may not be strictly convex. We also illustrate that the maximal solution of the dual problem may fail to have full support. Proofs are given in Section 4.

2 Statement of main results

As Kramkov and Schachermayer [25, 26], we assume that the utility function of the investor is a strictly increasing and strictly concave function $U : (0,\infty) \to \mathbb{R}$, which also is continuously differentiable and satisfies the Inada conditions

$$U'(0+) = +\infty \qquad \text{and} \qquad U'(\infty-) = 0.$$

Payoffs are modeled as random variables X on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Their utility shall be assessed in terms of a robust utility functional of the form

$$X \longmapsto \inf_Q \big(E_Q[\,U(X)\,] + \gamma(Q)\big). \tag{3}$$

Here we assume that γ is bounded from below and equal to the minimal penalty function of the convex risk measure ρ defined by

$$\rho(Y) := \sup_{Q \ll \mathbb{P}} \big(E_Q[\,-Y\,] - \gamma(Q) \big), \qquad Y \in L^\infty(\mathbb{P}).$$

That is, γ satisfies the biduality relation

$$\gamma(Q) = \sup_{Y \in L^\infty(\mathbb{P})} \big(E_Q[\,-Y\,] - \rho(Y) \big); \tag{4}$$

see [16, 18]. We may assume without loss of generality that ρ is normalized in the sense that $\rho(0) = -\inf_Q \gamma(Q) = 0$. We also assume the following conditions:

Assumption 2.1 *The risk measure ρ is continuous from below: If $Y_n \in L^\infty$ increases a.s. to $Y \in L^\infty$, then $\rho(Y_n) \to \rho(Y)$. It is also sensitive*[1] *in the sense that every nonzero $Y \in L^\infty_-$ satisfies $\rho(Y) > 0$.*

In Section 3 we have collected a number of particular examples for economically and statistically meaningful choices for γ. If γ only takes the values 0 and $+\infty$, then (3) reduces to the representation of a robust utility functional in the sense of Gilboa and Schmeidler [20]:

$$X \longmapsto \inf_{Q \in \mathcal{Q}} E_Q[\,U(X)\,] \tag{5}$$

for a convex set $\mathcal{Q}$ of probability measures. In this case, Assumption 2.1 is equivalent to [35, Assumption 2.1], as can be seen by combining the general representation theory of convex and coherent risk measures [18] with [35, Lemma 3.2] and Lemma 4.1 below. Even when restricted to this special case, our results will be stronger than those obtained in [35]. Particular examples for optimal investment problems with robust utility functionals of type (5) were analyzed by Quenez [30] and the author [32].

Remark 2.2 If the utility function U is not bounded from below, we must be careful in defining the expression $\inf_Q \big(E_Q[\,U(X)\,] + \gamma(Q)\big)$. First, it is clear that probabilistic models with an infinite penalty $\gamma(Q)$ should not contribute to the value of the robust utility functional. We therefore restrict the infimum to models Q in the domain

$$\mathcal{Q} := \{ Q \ll \mathbb{P} \,|\, \gamma(Q) < \infty \}$$

of γ. That is, we precise (3) by writing

$$X \longmapsto \inf_{Q \in \mathcal{Q}} \big(E_Q[\,U(X)\,] + \gamma(Q) \big).$$

Second, we have to address the problem that the Q-expectation of $U(X)$ might not be well-defined in the sense that $E_Q[\,U^+(X)\,]$ and $E_Q[\,U^-(X)\,]$ are both infinite. This problem will be resolved by extending the expectation operator $E_Q[\,\cdot\,]$ to the entire set L^0:

$$E_Q[\,F\,] := \sup_n E_Q[\,F \wedge n\,] = \lim_{n \uparrow \infty} E_Q[\,F \wedge n\,] \qquad \text{for arbitrary } F \in L^0. \tag{6}$$

It is easy to see that in doing so we retain the concavity of the functional $X \mapsto E_Q[\,U(X)\,]$ and hence of the robust utility functional.

[1] Sensitivity is also called *relevance*.

For the financial market model, we use the same setup as Kramkov and Schachermayer [25, 26]. The discounted price process of d assets is modeled by a stochastic process $S = (S_t)_{0\le t\le T}$. We assume that S is a d-dimensional semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. A self-financing trading strategy can be regarded as a pair (x, ξ), where $x \in \mathbb{R}$ is the initial investment and $\xi = (\xi_t)_{0\le t\le T}$ is a d-dimensional predictable and S-integrable process. The value process X associated with (x, ξ) is given by $X_0 = x$ and

$$X_t = X_0 + \int_0^t \xi_r \, dS_r \,, \qquad 0 \le t \le T \,.$$

For $x > 0$ given, we denote by $\mathcal{X}(x)$ the set of all value processes X that satisfy $X_0 \le x$ and are admissible in the sense that $X_t \ge 0$ for $0 \le t \le T$. We assume that our model is arbitrage-free in the sense that $\mathcal{M} \neq \emptyset$, where $\mathcal{M}$ denotes the set of measures equivalent to $\mathbb{P}$ under which each $X \in \mathcal{X}(1)$ is a local martingale; see [25]. Thus, our main problem can be stated as follows:

$$\text{Maximize } \inf_{Q\in\mathcal{Q}} \big(E_Q[\, U(X_T)\,] + \gamma(Q) \big) \text{ among all } X \in \mathcal{X}(x).$$

Consequently, the *value function of the robust problem* is defined as

$$u(x) := \sup_{X\in\mathcal{X}(x)} \inf_{Q\in\mathcal{Q}} \big(E_Q[\, U(X_T)\,] + \gamma(Q) \big).$$

One of our first results will be the minimax identity

$$u(x) = \inf_{Q\in\mathcal{Q}} \big(u_Q(x) + \gamma(Q) \big), \qquad \text{where} \qquad u_Q(x) := \sup_{X\in\mathcal{X}(x)} E_Q[\, U(X_T)\,].$$

The function u_Q is the value function of the optimal investment problem for an investor with subjective measure $Q \in \mathcal{Q}$. Next, we define as usual the convex conjugate function V of U by

$$V(y) := \sup_{x>0} \big(U(x) - xy \big), \qquad y > 0.$$

With this notation, it was stated in Theorem 3.1 of [25] that, for $Q \sim \mathbb{P}$ with finite value function u_Q,

$$u_Q(x) = \inf_{y>0} \big(v_Q(y) + xy \big) \qquad \text{and} \qquad v_Q(y) = \sup_{x>0} (u_Q(x) - xy), \tag{7}$$

where the dual value function v_Q is given by

$$v_Q(y) = \inf_{Y\in\mathcal{Y}_Q(y)} E_Q[\, V(Y_T)\,], \qquad Q \in \mathcal{Q},$$

and the space $\mathcal{Y}_Q(y)$ is defined as the set of all positive Q-supermartingales such that $Y_0 = y$ and XY is a Q-supermartingale for all $X \in \mathcal{X}(1)$. We thus define the *dual value function of the robust problem* by

$$v(y) := \inf_{Q\in\mathcal{Q}} \big(v_Q(y) + \gamma(Q) \big) = \inf_{Q\in\mathcal{Q}} \inf_{Y\in\mathcal{Y}_Q(y)} \big(E_Q[\, V(Y_T)\,] + \gamma(Q) \big).$$

Definition 2.3 Let $y > 0$ be such that $v(y) < \infty$. A pair (Q, Y) is a *solution of the dual problem* if $Q \in \mathcal{Q}$, $Y \in \mathcal{Y}_Q(y)$, and $v(y) = E_Q[\,V(Y_T)\,] + \gamma(Q)$.

Let us finally introduce the set $\mathcal{Q}_e$ of measures in $\mathcal{Q}$ that are equivalent to $\mathbb{P}$:

$$\mathcal{Q}_e := \{Q \in \mathcal{Q} \,|\, Q \sim \mathbb{P}\}.$$

Our assumptions on γ guarantee that $\mathcal{Q}_e$ is always nonempty; see Lemma 4.1.

Theorem 2.4 *In addition to the above assumptions, let us assume that*

$$u_{Q_0}(x) < \infty \text{ for some } x > 0 \text{ and some } Q_0 \in \mathcal{Q}_e \tag{8}$$

and that

$$v(y) < \infty \text{ implies } v_{Q_1}(y) < \infty \text{ for some } Q_1 \in \mathcal{Q}_e. \tag{9}$$

Then the robust value function u is concave, takes only finite values, and satisfies

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \big(E_Q[\,U(X_T)\,] + \gamma(Q)\big) = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} \big(E_Q[\,U(X_T)\,] + \gamma(Q)\big).$$

Moreover, the two robust value functions u and v are conjugate to another:

$$u(x) = \inf_{y>0} \big(v(y) + xy\big) \qquad \text{and} \qquad v(y) = \sup_{x>0} \big(u(x) - xy\big). \tag{10}$$

In particular, v is convex. The derivatives of u and v satisfy

$$u'(0+) = \infty \qquad \text{and} \qquad v'(\infty-) = 0. \tag{11}$$

If furthermore $v(y) < \infty$, then the dual problem admits a solution $(\widehat{Q}, \widehat{Y})$ that is maximal *in the sense that any other solution (Q, Y) satisfies $Q \ll \widehat{Q}$ and $Y_T = \widehat{Y}_T$ Q-a.s.*

It is possible that the maximal $\widehat{Q}$ is *not* equivalent to $\mathbb{P}$; see Example 3.2 below. If this happens, then $\widehat{Q}$ considered as a financial market model on its own may admit arbitrage opportunities. In this light, one also has to understand the conditions (8) and (9): They exclude the possibility that the value functions u_Q and v_Q are only finite for some degenerate model $Q \in \mathcal{Q}$, for which the duality relations (7) need not hold.

The situation simplifies considerably if we assume that *all* measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. In this case, condition (9) is always satisfied and (8) can be replaced by the assumption that $u(x) < \infty$ for some $x > 0$. Moreover, the optimal $\widehat{Y}$ is then $\mathbb{P}$-almost surely unique. Despite this fact, however, and in contrast to the situation in [25, 26], it can happen that the dual value function v is not strictly convex—even if all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. Equivalently, the value function u may fail to be continuously differentiable. This fact will be illustrated in Example 3.1 below. A sufficient condition for the strict convexity of v and the continuous differentiability of u is given in the next result. It applies in particular to entropic penalties (Example 3.3) and to penalty functions defined in terms of many other statistical distance functions (Example 3.8).

Proposition 2.5 *Suppose that the assumptions of Theorem 2.4 are satisfied and γ is strictly convex on $\mathcal{Q}$. Then u is continuously differentiable and v is strictly convex on its domain.*

Our next aim is to get existence results for optimal strategies. In the classical case $\mathcal{Q} = \{\mathbb{P}\}$, it was shown in [26] that a necessary and sufficient condition for the existence of optimal strategies at each initial capital is the finiteness of the dual value function $v_{\mathbb{P}}$. This condition translates as follows to our robust setting:

$$v_Q(y) < \infty \qquad \text{for all } y > 0 \text{ and each } Q \in \mathcal{Q}_e. \tag{12}$$

Recall from [26, Note 2] that (12) holds as soon as u_Q is finite for all $Q \in \mathcal{Q}_e$ and the asymptotic elasticity of the utility function U is strictly less than one:

$$AE(U) = \limsup_{x \uparrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Theorem 2.6 *In addition to the assumptions of Theorem 2.4, let us assume* (12). *Then both value functions u and v take only finite values and satisfy*

$$u'(\infty-) = 0 \qquad \textit{and} \qquad v'(0+) = -\infty. \tag{13}$$

The robust value function u is strictly concave, and the dual value function v is continuously differentiable. Moreover, for any $x > 0$ there exist an optimal strategy $\widehat{X} \in \mathcal{X}(x)$ for the robust problem. If $y > 0$ is such that $v'(y) = -x$ and $(\widehat{Q}, \widehat{Y})$ is a solution of the dual problem, then

$$\widehat{X}_T = I(\widehat{Y}_T) \qquad \widehat{Q}\text{-a.s.} \tag{14}$$

for $I := -V'$ and $(\widehat{Q}, \widehat{X})$ is a saddlepoint for the robust problem:

$$u(x) = \inf_{Q \in \mathcal{Q}} \big(E_Q[\,U(\widehat{X}_T)\,] + \gamma(Q) \big) = E_{\widehat{Q}}[\,U(\widehat{X}_T)\,] + \gamma(\widehat{Q}) = u_{\widehat{Q}}(x) + \gamma(\widehat{Q}).$$

Furthermore, $\widehat{X}\widehat{Y}\widehat{Z}$ is a martingale under $\mathbb{P}$, where $(\widehat{Z}_t)_{0 \le t \le T}$ is the density process of $\widehat{Q}$ with respect to $\mathbb{P}$.

In the preceding theorem, let us take $(\widehat{Q}, \widehat{Y})$ as a maximal solution of the dual problem as constructed in Theorem 2.4. Then the solution $\widehat{X}_T$ will be $\mathbb{P}$-a.s. unique as soon as $\widehat{Q} \sim \mathbb{P}$. This equivalence holds trivially if all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. In the general case, however, Example 3.2 will show that $\widehat{Q}$ need not be equivalent to $\mathbb{P}$, so that (14) cannot guarantee the $\mathbb{P}$-a.s. uniqueness of $\widehat{X}_T$. Nevertheless, we can construct an optimal strategy from a given solution of the dual problem by superhedging an appropriate contingent claim $H \ge 0$:

Corollary 2.7 *Suppose the assumptions of Theorem 2.6 hold. Let $(\widehat{Q}, \widehat{Y})$ be a solution of the dual problem at level $y > 0$ and consider the contingent claim*

$$H := I(\widehat{Y}_T) \mathrm{I}_{\{\widehat{Z} > 0\}},$$

where $d\widehat{Q} = \widehat{Z}\,d\mathbb{P}$. Then $x = -v'(y)$ is the minimal initial investment $x' > 0$ for which there exists some $X \in \mathcal{X}(x')$ such that $X_T \geq H$ $\mathbb{P}$-a.s. If furthermore $\widehat{X} \in \mathcal{X}(x)$ is such a strategy, then it is a solution for the robust utility maximization problem at initial capital x.

Remark 2.8 Instead of working with the terminal values of processes in the space $\mathcal{Y}_Q(y)$, it is sometimes more convenient to work with the densities of measures in the set $\mathcal{M}$ of equivalent local martingale measures. In fact, the dual value function satisfies

$$v(y) = \inf_{P^* \in \mathcal{M}} \inf_{Q \in \mathcal{Q}_e} \Big(E_Q\Big[V\Big(y \frac{dP^*}{dQ} \Big) \Big] + \gamma(Q) \Big). \tag{15}$$

This identity follows from Lemma 4.4 below and the corresponding identity in [25, 26]. Since the infimum in (15) need not be attained, it is often not possible to represent the optimal solution $\widehat{X}_T$ in terms of the density of an equivalent martingale measure. Nevertheless, Föllmer and Gundel [15] recently observed that the elements of $\mathcal{Y}_Q(1)$ can be interpreted as density processes of 'extended martingale measures'.

3 Examples and Counterexamples

The first example in this section illustrates that the value function u need not be continuously differentiable and its dual v need not be strictly convex, even if all measures in $\mathcal{Q}$ are mutually equivalent. The second example illustrates that the maximal solution of the dual problem, as constructed in Theorem 2.4, may not have full support. The subsequent examples provide explicit choices for penalty functions γ, which are natural from an economical or statistical point of view. They will also illustrate that control methods are often not feasible for robust optimization problems.

Example 3.1 (Non-differentiability of the value function) We consider a one-period trinomial model where the risky asset starts off at $S_0 = 1$. At time $t = 1$, it can take the values 0, 1, and 2. Consequently, we let $\Omega := \{\omega_-, \omega_0, \omega_+\}$ and define $S_1(\omega_\pm) := 1 \pm 1$ and $S_1(\omega_0) := 1$. A probability measure Q on Ω is determined by $p := Q[\{\omega_+\}]$ and $q := Q[\{\omega_-\}]$. This model fits into the semimartingale framework by taking $S_t := 1$ and $\mathcal{F}_t := \{\emptyset, \Omega\}$ as long as $t < 1$ and $\mathcal{F}_1 := \sigma(S_1)$. It is arbitrage-free and satisfies the assumption $\mathcal{M} \neq \emptyset$ iff p and q are both strictly positive. An investment ξ in the risky asset made for an initial wealth x results in a terminal payoff $X_1 = x + \xi(S_1 - S_0)$. Hence, ξ is admissible iff $|\xi| \leq x$. Let us take $U(x) = \sqrt{x}$. Then the Q-expected utility of an admissible investment ξ is given by

$$E_Q\big[U(x + \xi(S_1 - S_0)) \big] = p\sqrt{x + \xi} + (1 - p - q)\sqrt{x} + q\sqrt{x - \xi}.$$

Optimizing over ξ yields that

$$\xi = x \cdot \frac{p^2 - q^2}{p^2 + q^2}$$

is the unique optimal strategy for Q. Now we take $0 < a < b < 2/3$ and define $\mathcal{Q}$ as the set of all measures Q_p for which $q = p/2$ and $a \le p \le b$. This set $\mathcal{Q}$ is parameterized by p and consists of mutually equivalent measures. For $Q_p \in \mathcal{Q}$, the value function is given by

$$u_{Q_p}(x) = \sqrt{x}(1 + \beta p),$$

where $\beta = \sqrt{8/5} - 3/2 + 1/\sqrt{10} > 0$. The penalty function

$$\gamma(Q) := \begin{cases} \beta(b-p) & \text{if } Q = Q_p \in \mathcal{Q}, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex and lower semicontinuous and thus satisfies our assumptions. By Theorem 2.4, the robust value function is given by

$$u(x) = \inf_{Q\in\mathcal{Q}} \big(u_Q(x) + \gamma(Q)\big) = \sqrt{x} + \beta b + \beta \inf_{a\le p\le b} \big(p\sqrt{x} - p\big).$$

The infimum on the right equals $b\sqrt{x} - b$ for $x < 1$ and $a\sqrt{x} - a$ for $x > 1$. Hence, u is not continuously differentiable at $x = 1$, and v cannot be strictly convex; see, e.g., [31, Theorem V.26.3].

Example 3.2 (The maximal $\widehat{\mathbf{Q}}$ may fail to have full support) The fact that the measure $\widehat{Q}$ associated with the maximal solution for the dual problem may not be equivalent to $\mathbb{P}$ can be deduced from [35, Example 2.5 and Theorem 2.6]. Here we give a more direct argument within the setting of [35, Example 2.5]. We consider a one-period model in discrete time ($t = 0, 1$) with two risky assets S^1, S^2 satisfying $S^1_0 = S^2_0 = 1$. Under the measure $\mathbb{P} := Q_1$, the first asset S^1_1 has the distribution

$$Q_1[\,S^1_1 = 2\,] =: q = 1 - Q_1[\,S^1_1 = 0\,],$$

where $1/2 < q < 1$. The second asset S^2_1 has support $\{0, 1, \dots\}$, and finite expected value $E_{Q_1}[\,S^2_1\,] > S^2_0 = 1$. We take $\mathbb{P} := Q_1$ as our reference measure. We introduce another measure $Q_0 \ll \mathbb{P}$ by requiring that

$$Q_0[\,S^1_1 = 2\,] = Q_0[\,S^1_1 = 0\,] = 1/2 \qquad \text{and} \qquad Q_0[\,S^2_1 = 0\,] = 1.$$

We define $\gamma(Q) = 0$ if $Q = Q_\alpha := \alpha Q_1 + (1-\alpha)Q_0$ for some $0 \le \alpha \le 1$ and $\gamma(Q) = \infty$ otherwise. Note that a trading strategy can only be admissible for $\mathbb{P} = Q_1$ if it does not contain short positions in the second asset, because S^2_1 is unbounded. Hence, under Q_0 any strategy $X \in \mathcal{X}(1)$ is a supermartingale, and it follows that $v_{Q_0}(y) = V(y)$. Under Q_α with $\alpha > 0$, any long position in the first asset will be a submartingale, and so we must have $v_{Q_\alpha}(y) > V(y)$. This shows that $\widehat{Q} = Q_0$ and $\widehat{Y}_1 = y\mathrm{I}_{\{S^2_1=0\}}$ is the unique solution of the dual problem. Moreover, one can easily show that $\widehat{X}_1 \equiv x$ is the unique solution of the primal problem; see [35, Example 2.5]. The constant $y = I(x)$, however, does not belong to any of the spaces $\mathcal{Y}_{Q_\alpha}(y)$ for $\alpha > 0$. This illustrates that it is possible that the duality relation

$$\widehat{X}_T = I(\widehat{Y}_T)$$

cannot be extended to a $\mathbb{P}$-a.s. identity. Finally, note that Q_0 considered as a market model on its own has not the same admissible strategies as $\mathbb{P}$, since short selling the second asset is admissible in the model Q_0. In fact, such short sales even create arbitrage opportunities under Q_0. $\diamondsuit$

Example 3.3 (Entropic penalties) A popular choice for γ is taking (a multiple of) the relative entropy with respect to $\mathbb{P}$, which is defined as

$$H(Q|\mathbb{P}) = \int \frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}}\, d\mathbb{P} = \sup_{Y \in L^\infty} \big(E_Q[\, Y\,] - \log \mathbb{E}[\, e^Y\,] \big), \qquad Q \ll \mathbb{P};$$

see, e.g., [18, Sections 3.2 and 4.9]. Due to the classical duality formula

$$\log \mathbb{E}[\, e^X\,] = \sup_{Q \in \mathcal{Q}} \big(E_Q[\, X\,] - H(Q|\mathbb{P}) \big), \tag{16}$$

the choice $\gamma(Q) = \frac{1}{\theta} H(Q|\mathbb{P})$ corresponds to the utility functional

$$\inf_{Q \in \mathcal{Q}} \big(E_Q[\, U(X_T)\,] + \gamma(Q) \big) = -\frac{1}{\theta} \log \mathbb{E}\big[\, e^{-\theta U(X_T)}\,\big]$$

of the terminal wealth, which obviously satisfies Assumption 2.1. Its maximization is equivalent to the maximization of the ordinary expected utility $\mathbb{E}[\,\widetilde{U}(X_T)\,]$, where $\widetilde{U}(x) = -e^{-\theta U(x)}$ is strictly concave, increasing, and satisfies the Inada conditions. Thus, robustness effects are only felt in *intertemporal* optimization problems; see Hansen and Sargent [21], Barrieu and El Karoui [2], or Bordigoni et al. [4]. For related problems see, e.g., El Karoui et al [13], Schroder and Skiadas [37], and the references therein.

The use of entropic penalties in intertemporal optimization problems is facilitated by the *dynamic consistency* of the corresponding conditional risk measure $\rho_t(X) := \frac{1}{\theta} \log \mathbb{E}[\, e^{-\theta X} \mid \mathcal{F}_t\,]$, namely,

$$\rho_0(-\rho_t(Y)) = \rho_0(Y) \qquad \text{for all } Y \in L^\infty. \tag{17}$$

This property of dynamic consistency corresponds to the Bellman principle in dynamic programming and is the essential ingredient for the application of control methods.

Example 3.4 (A class of dynamically consistent penalties) Suppose that the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ is generated by a d-dimensional standard Brownian motion W. Then for every measure $Q \ll \mathbb{P}$ there exists a d-dimensional predictable process η such that $\int_0^T |\eta_t|^2\, dt < \infty$ Q-a.s. and $dQ/d\mathbb{P} = \mathcal{E}(\int_0 \eta_t\, dW_t)_T$ Q-a.s., where $\mathcal{E}(M)_t = \exp(M_t - \langle M \rangle_t)$ denotes the stochastic exponential of a continuous semimartingale M. Let $h : \mathbb{R} \to [0, \infty]$ be a lower semicontinuous proper convex function and suppose that there are constants $\kappa_1, \kappa_2 > 0$ such that $h(x) \ge \kappa_1 |x|^2 - \kappa_2$. Then the penalty functions

$$\gamma_t(Q) := E_Q\Big[\int_t^T h(\eta_u)\, du \,\Big|\, \mathcal{F}_t \Big]$$

define a dynamically consistent family $(\rho_t)_{0 \le t \le T}$ of risk measures. Moreover, γ_0 is the minimal penalty function of ρ_0, and ρ_0 satisfies Assumption 2.1; see [23, Lemma 4.1]. Note that the case $h(x) = \frac{1}{2}|x|^2$ corresponds to the entropic penalty function $\gamma(Q) = H(Q|\mathbb{P})$.

Remark 3.5 Recently, the dynamic consistency (17) of risk measures has been the subject of intense study; see, e.g., [7], [12], and the references therein. As explained above, it is the crucial property needed for an application of control methods and thus greatly facilitates computations. As a normative economic postulate, however, it is debatable as it would require that the investor does not change the penalty function for the entire investment period $[0, T]$ (apart from the obvious Markovian-type updating). But financial models are typically not accurate, and each piece of freshly revealed information might require to adjust models and hence penalty functions. In reality, this fact is usually taken into account by a periodic model recalibration, resulting in ever changing model parameters and thus non-Markovian updating.

In addition to the argument in the preceding remark, the following examples will illustrate that some natural risk measures do not satisfy the property (17). These examples all belong to the class of law-invariant convex risk measures. The failure of dynamic consistency for law-invariant *coherent* risk measures has already been pointed out by Delbaen [10]. In [32] one can find number of explicit computations of optimal strategies for robust utility functionals defined in terms of coherent risk measures for which dynamic consistency (17) is not given. The method in [32], however, is confined to complete market models, whereas our duality results presented in Section 2 are available in the general case.

Example 3.6 (Shortfall risk) Let $\ell : \mathbb{R} \to \mathbb{R}$ be convex, increasing, and nonconstant and take x in the interior of $\ell(\mathbb{R})$. The associated *shortfall risk measure*

$$\rho(Y) := \inf\big\{ m \in \mathbb{R} \,|\, \mathbb{E}[\,\ell(-Y-m)\,] \le x \big\}, \qquad Y \in L^\infty, \tag{18}$$

was introduced by Föllmer and the author in [16]. Assumption 2.1 is satisfied due to [18, Proposition 4.104]. Using (16), one sees that the choice $\ell(y) = e^{\theta y}$ corresponds to the entropic penalty $\gamma(Q) = \frac{1}{\theta}H(Q|\mathbb{P})$. For general ℓ, the penalty function is given by

$$\gamma(Q) = \inf_{\lambda>0} \frac{1}{\lambda}\Big(x + \mathbb{E}\Big[\,\ell^*\Big(\lambda\frac{dQ}{d\mathbb{P}}\Big)\,\Big]\Big) = \inf_{\lambda>0}\Big(\lambda x + \lambda\mathbb{E}\Big[\,\ell^*\Big(\lambda^{-1}\frac{dQ}{d\mathbb{P}}\Big)\,\Big]\Big), \qquad Q \ll \mathbb{P}, \tag{19}$$

where ℓ^* is the Fenchel-Legendre transform of ℓ; see [16, Theorem 10] or [18, Theorem 4.106]. The risk measure ρ satisfies $\rho(0) = 0$ if we take $x = \ell(0)$. It induces a dynamic risk measure ρ_t, $0 \le t \le T$, in a canonical way by replacing the expectation operator in (18) with a conditional expectation. It is easy to see that this dynamic risk measure is *weakly dynamically consistent* in the sense that

$$\rho_t(Y) \le 0 \text{ } \mathbb{P}\text{-a.s.} \ \Rightarrow \ \rho_0(Y) \le 0 \qquad \text{and} \qquad \rho_t(Y) \ge 0 \text{ } \mathbb{P}\text{-a.s.} \ \Rightarrow \ \rho_0(Y) \ge 0; \tag{20}$$

see Weber [39]. This weak property, however, does not guarantee the validity of (17), as is illustrated by the following simple example.

Example 3.7 (Shortfall risk may not be dynamically consistent) As a loss function we take $\ell(y) = (y+\varepsilon)^+$, where $0 < \varepsilon < 1/4$. Let Y_1, Y_2 be two Bernoulli random variables such that $\mathbb{P}[\,Y_i = 0\,] = \mathbb{P}[\,Y_i = 1\,] = 1/2$, $i = 1, 2$. Suppose that Y_1 is $\mathcal{F}_1$-measurable and Y_2 is independent of $\mathcal{F}_1$, while $\mathcal{F}_0$ is trivial. We let $Y := -Y_1Y_2$ and compute its risk under the dynamic shortfall risk measure arising from (18), which is normalized if we choose $x = \varepsilon$. A straightforward computation then shows that $\rho_0(Y) = 1 - 3\varepsilon$, while $\rho_1(Y) = (1-\varepsilon)Y_1$ and $\rho_0(-\rho_1(Y)) = 1 - 2\varepsilon$. We believe that this failure of dynamic consistency for shortfall risk is the rule rather than the exception.

Note the the condition of weak dynamic consistency (20) is necessary for (17). Yet, Weber [39] showed that, under certain technical regularity conditions, shortfall risk is the only law-invariant risk measure such that the associated canonical dynamic risk measure is weakly dynamically consistent. Here is another natural choice for a law-invariant risk measure, which may not even satisfy (20).

Example 3.8 (Statistical distance functions) Let $g : [0,\infty) \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous convex function satisfying $g(1) < \infty$ and the superlinear growth condition $g(x)/x \to +\infty$ as $x \uparrow \infty$. Associated to it is the *g-divergence*

$$I_g(Q|\mathbb{P}) := \mathbb{E}\Big[\, g\Big(\frac{dQ}{d\mathbb{P}}\Big)\Big], \qquad Q \ll \mathbb{P},$$

as introduced by Csiszar [8, 9]. The g-divergence $I_g(Q|\mathbb{P})$ can be interpreted as a statistical distance between the hypothetical model Q and the reference measure $\mathbb{P}$, so that taking $\gamma(Q) := I_g(Q|\mathbb{P})$ is a natural choice for a penalty function. The particular choice $g(x) = x\log x$ corresponds to the relative entropy $I_g(Q|\mathbb{P}) = H(Q|\mathbb{P})$. Taking $g(x) = 0$ for $x \le \lambda^{-1}$ and $g(x) = \infty$ otherwise corresponds to the coherent risk measure Average Value at Risk,

$$AVaR_\lambda(Y) = \sup\big\{ E_Q[\,-Y\,] \,|\, dQ/d\mathbb{P} \le \lambda^{-1} \big\},$$

which is also called Expected Shortfall or Conditional Value at Risk. One easily sees that $AVaR_\lambda$ does *not* satisfy the condition of weak dynamic consistency (20); see also [39]. In particular it does not satisfy (17). See [32, 33] for an analysis of optimal investment problems for $AVaR_\lambda$ in complete market models.

For general g the penalty function $\gamma(Q) = I_g(Q|\mathbb{P})$ corresponds to the convex risk measure

$$\rho(Y) = \sup_{Q\ll\mathbb{P}} \big(E_Q[\,-Y\,] - \gamma(Q)\big), \qquad Y \in L^\infty,$$

which satisfies Assumption 2.1 and will be normalized as soon as $g(1) = 0$. Indeed, the level sets $\{dQ/d\mathbb{P} \,|\, I_g(Q|\mathbb{P}) \le c\}$ are weakly compact in $L^1(\mathbb{P})$ due to the superlinear growth condition, and so continuity from below follows from [28, Lemma 2] together with [18, Corollary 4.35]; see also [24, Theorem 2.4]. The convex risk measure ρ satisfies the variational identity

$$\rho(Y) = \sup_{Q\ll\mathbb{P}} \big(E_Q[\,-Y\,] - \gamma(Q)\big) = \inf_{z\in\mathbb{R}} \big(\mathbb{E}[\,g^*(z-Y)\,] - z\big), \qquad Y \in L^\infty, \tag{21}$$

where $g^*(y) = \sup_{x>0}(xy - g(x))$. In the particular situation considered in this example, the identity (21) yields an alternative way for transforming the original maximin problem of robust utility maximization into a simpler minimization problem; see also the arguments in [34, Theorem 2.14]. The formula (21) was obtained by Ben-Tal and Teboulle [3] for $\mathbb{R}$-valued g. In the case of $AVaR_\lambda$, we have $g^*(y) = 0 \vee y/\lambda$ and hence recover [18, Lemma 4.46] as a special case of (21). Below we will give a proof, which works in the general case and is based on the results from Föllmer and the author [16, 18] quoted in Example 3.6.

Proof of (21): For $\lambda > 0$ let $g_\lambda(x) := \lambda g(x/\lambda)$. Then $(\lambda, x) \mapsto g_\lambda(x)$ is convex due to (25) below. Let $\gamma_\lambda(Q) = I_{g_\lambda}(Q|\mathbb{P})$ be the corresponding g_λ-divergence. Then $(\lambda, Q) \mapsto \gamma_\lambda(Q)$ is a convex functional, and it follows easily that

$$h(\lambda) := \begin{cases} \min_{Q \ll \mathbb{P}} \big(E_Q[\,Y\,] + \gamma_\lambda(Q) \big) & \text{if } \lambda > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

is a lower semicontinuous convex function in λ if $Y \in L^\infty$ is fixed. Our aim is to compute $h(1)$. The idea is to use the fact that (19) is the penalty function of the risk measure in (18) in order to identify the Fenchel-Legendre transform h^* of h. We only have to observe that $\ell := g^*$ satisfies the assumptions of Example 3.6 and that $\ell^* = g^{**} = g$ so as to apply (19):

$$\begin{aligned} f(x) &:= \inf \big\{ \, m \in \mathbb{R} \,|\, \mathbb{E}[\, g^*(-m - Y)\,] \le x \, \big\} \\ &= \sup_{Q \ll \mathbb{P}} \Big(E_Q[\,-Y\,] - \inf_{\lambda > 0} \Big(\lambda x + \mathbb{E}\Big[\, g_\lambda\Big(\frac{dQ}{d\mathbb{P}}\Big)\,\Big]\Big)\Big) \\ &= -\inf_{\lambda>0} \inf_{Q \ll \mathbb{P}} \Big(E_Q[\,Y\,] + \lambda x + \gamma_\lambda(Q) \Big) \\ &= -\inf_{\lambda > 0} \big(\lambda x + h(\lambda) \big) = h^*(-x), \end{aligned}$$

for all x in the interior of $g^*(\mathbb{R})$, which coincides with the interior of $\operatorname{dom} f$. Convexity hence yields $h(1) = h^{**}(1) = \sup_x (x - f(-x))$. The definition of f yields that $x = -\mathbb{E}[\, g^*(-f(-x) - Y)\,]$. Hence,

$$h(1) = \sup_{x \in \mathbb{R}} \big(- \mathbb{E}[\, g^*(-f(-x) - Y)\,] - f(-x) \big),$$

and the assertion follows by noting that the range of f contains all points to the left of $\|Y^-\|_\infty - x_0$, where x_0 is the lower bound all points in which the right-hand derivative of g^* is strictly positive. □

4 Proofs

For $c \ge 0$, let us introduce the sets

$$\mathcal{Q}(c) := \{ Q \in \mathcal{Q} \,|\, \gamma(Q) \le c \} \qquad \text{and} \qquad \mathcal{Q}_e(c) := \{ Q \in \mathcal{Q}(c) \,|\, Q \sim \mathbb{P} \}.$$

With $\mathcal{Z}(c)$, $\mathcal{Z}$, $\mathcal{Z}_e(c)$, and $\mathcal{Z}_e$, we will denote the corresponding sets of densities, e.g.,

$$\mathcal{Z} := \Big\{ \frac{dQ}{d\mathbb{P}} \,\big|\, Q \in \mathcal{Q} \Big\}, \qquad \mathcal{Z}_e(c) := \Big\{ \frac{dQ}{d\mathbb{P}} \,\big|\, Q \in \mathcal{Q}_e(c) \Big\}.$$

In the sequel, we will identify measures $Q \in \mathcal{Q}$ with their densities $Z = dQ/d\mathbb{P}$, and we will also write $\gamma(Z)$, u_Z, v_Z for $\gamma(Q)$, u_Q, and v_Q, respectively. Due to (4), $Z \mapsto \gamma(Z)$ is a convex and weakly lower semicontinuous functional on $L^1(\mathbb{P})$.

Lemma 4.1 *For every $c > 0$, the level set $\mathcal{Z}(c)$ is weakly compact, and $\mathcal{Z}_e(c)$ is nonempty. Moreover, $Z \mapsto \gamma(Z)$ is lower semicontinuous with respect to $\mathbb{P}$-a.s. convergence on $\mathcal{Z}(c)$.*

Proof: The set $\mathcal{Z}(c)$ is weakly closed by the weak lower semicontinuity of γ and uniformly integrable due to [18, Lemma 4.22]. Hence, $\mathcal{Z}(c)$ is weakly compact according to the Dunford-Pettis theorem. Next, for all $c > 0$ we have that $\mathbb{P}[\,A\,] > 0$ implies $Q[\,A\,] > 0$ for some $Q \in \mathcal{Q}(c)$. Indeed, the sensitivity of ρ gives

$$0 < \rho(-c\mathrm{I}_A) = \sup_{Q \in \mathcal{Q}(c)} (cQ[\,A\,] - \gamma(Q)).$$

Hence, the assertion $\mathcal{Z}_e(c) \neq \emptyset$ follows from the Halmos-Savage theorem. Finally, if $Z_n \to Z$ $\mathbb{P}$-a.s. and all Z_n belong to some level set $\mathcal{Z}(c)$, then convergence also holds in $L^1(\mathbb{P})$, and the lower semicontinuity of γ follows from (4). $\square$

We note next that the space $\mathcal{Y}_Q(y)$ can easily be related to $\mathcal{Y}(y) := \mathcal{Y}_{\mathbb{P}}(y)$:

Lemma 4.2 *Let $(Z_t)_{0 \le t \le T}$ be the density process of $Q \ll \mathbb{P}$ with respect to $\mathbb{P}$. Then a process Y^Q belongs to $\mathcal{Y}_Q(y)$ if and only if $Y^Q Z \in \mathcal{Y}(y)$. In particular, we have*

$$v(y) = \inf_{Z \in \mathcal{Z}} \inf_{Y \in \mathcal{Y}(y)} \Big(\mathbb{E}\Big[\, ZV\Big(\frac{Y_T}{Z}\Big) \Big] + \gamma(Z) \Big). \tag{22}$$

Proof: Take $0 \le s < t \le T$. If $Y^Q \in \mathcal{Y}_Q(y)$ and $X \in \mathcal{X}(1)$, then

$$X_s Y^Q_s \ge E_Q[\, X_t Y^Q_t \,|\, \mathcal{F}_s \,] = \frac{1}{Z_s} \mathbb{E}[\, X_t Y^Q_t Z_t \,|\, \mathcal{F}_s \,] \qquad \mathbb{P}\text{-a.s. on } \{Z_s > 0\}.$$

On $\{Z_s = 0\}$ we have $\mathbb{P}$-a.s. $Z_t = 0$ and hence $\mathbb{E}[\, X_t Y^Q_t Z_t \,|\, \mathcal{F}_s \,] = 0 = X_s Y^Q_s Z_s$. Combining these two facts shows that $XY^Q Z$ is a $\mathbb{P}$-supermartingale and hence that $Y^Q Z \in \mathcal{Y}(y)$. Conversely, suppose that $Y := Y^Q Z \in \mathcal{Y}(y)$. Then we have Q-a.s. for each $X \in \mathcal{X}(1)$

$$E_Q[\, X_t Y^Q_t \,|\, \mathcal{F}_s \,] = \frac{1}{Z_s} \mathbb{E}\big[\, X_t Y_t \,|\, \mathcal{F}_s \,\big] \le \frac{X_s Y_s}{Z_s} = X_s Y^Q_s.$$

$\square$

The formula (22) is more convenient than our original definition of v, as the infimum is now taken over two sets that are no longer related to another. As in [25, 26], we obtain "abstract versions" of our theorems if we replace the spaces $\mathcal{X}(x)$ and $\mathcal{Y}_Q(y)$ by the respective spaces

$$\mathcal{C}(x) = \big\{\, g \in L^0_+(\Omega, \mathcal{F}_T, \mathbb{P}) \,|\, 0 \le g \le X_T \text{ for some } X \in \mathcal{X}(x) \,\big\}.$$

and

$$\mathcal{D}_Q(y) = \big\{\, h \in L^0_+(\Omega, \mathcal{F}_T, Q) \,|\, 0 \le h \le Y_T \text{ for some } Y \in \mathcal{Y}_Q(y) \,\big\}.$$

Obviously, this substitution does not affect the values of our value functions, i.e., using our convention (6) we have $u_Q(x) = \sup_{g\in\mathcal{C}(x)} E_Q[\,U(g)\,]$, $v_Q(y) = \inf_{h\in\mathcal{D}_Q(y)} E_Q[\,V(h)\,]$,

$$u(x) = \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}} \big(E_Q[\,U(g)\,] + \gamma(Q)\big),$$

and

$$v(y) = \inf_{Q\in\mathcal{Q}} \inf_{h\in\mathcal{D}_Q(y)} \big(E_Q[\,V(h)\,] + \gamma(Q)\big) = \inf_{Z\in\mathcal{Z}} \inf_{h\in\mathcal{D}(y)} \Big(\mathbb{E}\Big[\,ZV\Big(\frac{h}{Z}\Big)\,\Big] + \gamma(Z)\Big).$$

Moreover, any optimal g or h, if they exist, can be taken as the terminal value of some process $X \in \mathcal{X}(x)$ or $Y \in \mathcal{Y}_Q(y)$. Next, recall from [25] that for $Q \sim \mathbb{P}$ and $x, y \ge 0$ given,

$$\begin{aligned} g \in \mathcal{C}(x) \quad &\Longleftrightarrow \quad g \ge 0 \text{ and } \sup_{h\in\mathcal{D}_Q(y)} E_Q[\,hg\,] \le xy \\ h \in \mathcal{D}_Q(y) \quad &\Longleftrightarrow \quad h \ge 0 \text{ and } \sup_{g\in\mathcal{C}(x)} E_Q[\,hg\,] \le xy. \end{aligned} \tag{23}$$

We point out that validity of this relation is not clear for $Q \not\sim \mathbb{P}$, and this will create a few technical difficulties in the sequel.

Under the convention (6), $g \mapsto E_Q[\,U(g)\,]$ is a concave functional on $\mathcal{C}(x)$ for each $Q \in \mathcal{Q}$ and all $x > 0$. Using the fact that

$$\{\, \alpha g + (1-\alpha)g' \,|\, g \in \mathcal{C}(x),\ g' \in \mathcal{C}(x') \,\} \subset \mathcal{C}(\alpha x + (1-\alpha)x')$$

then yields the concavity of the value functions u_Q and u. The concavity of u_Q implies in turn that

$$u_Q \equiv +\infty \text{ as soon as } E_Q[\,U^+(g)\,] = +\infty \text{ for some } g \in \bigcup_{x>0} \mathcal{C}(x); \tag{24}$$

see [35, Lemma 3.1].

A key observation for our future analysis is the convexity of the function $(z, y) \mapsto zV(y/z)$. In fact, one has

$$(\alpha z_0 + (1-\alpha)z_1)V\bigg(\frac{\alpha y_0 + (1-\alpha)y_1}{\alpha z_0 + (1-\alpha)z_1}\bigg) < \alpha z_0 V\Big(\frac{y_0}{z_0}\Big) + (1-\alpha)z_1 V\Big(\frac{y_1}{z_1}\Big) \tag{25}$$

as soon as $y_0/z_0 \ne y_1/z_1$ and $0 < \alpha < 1$; see Equation (21) in [35].

Lemma 4.3 *If $v(y) < \infty$, then there exist $\widehat{h} \in \mathcal{D}(y)$ and $\widehat{Z} \in \mathcal{Z}$ such that*

$$v(y) = \mathbb{E}[\,\widehat{Z}V(\widehat{h}/\widehat{Z})\,] + \gamma(\widehat{Z}).$$

Moreover, $\widehat{Z} =: d\widehat{Q}/d\mathbb{P}$ and $\widehat{h}$ can be chosen in such a way that $\widehat{h}/\widehat{Z}$ coincides $\widehat{Q}$-a.s. with the terminal value of some $\widehat{Y} \in \mathcal{Y}_{\widehat{Q}}(y)$ and such that $(\widehat{Q}, \widehat{Y})$ is a solution of the dual problem, which is maximal in the sense of Theorem 2.4.

Proof: Let $(Z_n, h_n) \in \mathcal{Z} \times \mathcal{D}(y)$ be a sequence such that $\mathbb{E}[\,Z_n V(h_n/Z_n)\,] + \gamma(Z_n) \to v(y)$. Jensen's inequality implies that

$$\mathbb{E}[\,ZV(h/Z)\,] \geq V\big(\mathbb{E}[\,h\,\mathrm{I}_{\{Z>0\}}\,]\big) \geq V(y) \qquad \text{for all } Z \text{ and } h \in \mathcal{D}(y). \tag{26}$$

Hence we must have $c := 1 + \limsup_n \gamma(Z_n) < \infty$, and so we can assume without loss of generality that $Z_n \in \mathcal{Z}(c)$ for all n.

Applying twice the standard Komlos-type argument of Lemma A1.1 in [11], we obtain a sequence

$$(\widetilde{Z}_n, \widetilde{h}_n) \in \mathrm{conv}\{(Z_n, h_n), (Z_{n+1}, h_{n+1}), \dots\} \subset \mathcal{Z}(c) \times \mathcal{D}(y)$$

that converges $\mathbb{P}$-a.s. to some $(\widehat{Z}_0, \widehat{h}_0)$. From (23) we get $\widehat{h}_0 \in \mathcal{D}(y)$. Lemma 4.1 implies $\widehat{Z}_0 \in \mathcal{Z}(c)$. It was shown in the proof of [35, Lemma 3.7] that the function

$$\mathcal{Z}(c) \times \mathcal{D}(y) \ni (Z, h) \longmapsto \mathbb{E}\big[\,ZV\big(h/Z\big)\,\big] \tag{27}$$

is lower semicontinuous with respect to $\mathbb{P}$-a.s. convergence. By the convexity of $(x, z) \mapsto zV(x/z)$ and Lemma 4.1 we thus get

$$\mathbb{E}[\,\widehat{Z}_0 V(\widehat{h}_0/\widehat{Z}_0)\,] + \gamma(\widehat{Z}_0) \leq \liminf_{n\uparrow\infty} \Big(\mathbb{E}[\,Z_n V(h_n/Z_n)\,] + \gamma(Z_n)\Big) = v(y).$$

In this sense, the pair $(\widehat{h}_0, \widehat{Z}_0)$ is optimal.

Suppose $(\widehat{h}_1, \widehat{Z}_1)$ is another optimal pair, and let $\widehat{h}_t := t\widehat{h}_1 + (1-t)\widehat{h}_0$ and $\widehat{Z}_t := t\widehat{Z}_1 + (1-t)\widehat{Z}_0$ for $0 \leq t \leq 1$. The convexity of $(h, Z) \mapsto \mathbb{E}[\,ZV(h/Z)\,] + \gamma(Z)$ implies that each pair $(\widehat{h}_t, \widehat{Z}_t)$ is also optimal. If $0 < t < 1$, then $\{\widehat{Z}_t > 0\} = \{\widehat{Z}_0 > 0\} \cup \{\widehat{Z}_1 > 0\}$. Moreover, (25) shows that the ratio $\widehat{h}_t/\widehat{Z}_t$ does not depend on $t \in (0, 1)$. Hence, there exists a random variable $\widehat{Y}_T \geq 0$ and a sequence $\widetilde{Z}_1, \widetilde{Z}_2, \dots$ such that the following hold:

(a) $\mathbb{P}[\,\widetilde{Z}_n > 0\,]$ tends to the maximum $\mathbb{P}$-probability for the support of any optimal $\widehat{Z}$;

(b) $\{\widetilde{Z}_1 > 0\} \subset \{\widetilde{Z}_2 > 0\} \subset \cdots$;

(c) for each n, we have $\widetilde{h}_n := \widehat{Y}_T \widetilde{Z}_n \in \mathcal{D}(y)$, and the pair $(\widetilde{h}_n, \widetilde{Z}_n)$ is optimal.

By using a Komlos-type argument, we may assume that the $\widetilde{Z}_n$ converge $\mathbb{P}$-a.s to some $\widehat{Z} \in \mathcal{Z}$. Then $\widehat{Y}_T \widehat{Z} \in \mathcal{D}(y)$ by (23) and in turn $\widehat{Y}_T \in \mathcal{D}_{\widehat{Q}}(y)$ due to Lemma 4.2. Hence, we may assume that $\widehat{Y}_T$ is the terminal value of some $\widehat{Y} \in \mathcal{Y}_{\widehat{Q}}(y)$. As above, we then conclude $E_{\widehat{Q}}[\,V(\widehat{Y}_T)\,] + \gamma(\widehat{Q}) \leq v(y)$, that is, $(\widehat{Q}, \widehat{Y})$ is a solution of the dual problem. Clearly, $(\widehat{Q}, \widehat{Y})$ is maximal. ☐

Let $\mathcal{Q}^f$ denote the set of $Q \in \mathcal{Q}$ such that $u_Q(x) < \infty$ for some and hence all $x > 0$. Similarly we define $\mathcal{Q}^f_e$, $\mathcal{Z}^f$, and $\mathcal{Z}^f_e$. We will show next that in (22) the set $\mathcal{Q}$ can be replaced by the smaller sets $\mathcal{Q}_e$ and $\mathcal{Q}^f_e$.

Lemma 4.4 *For $v(y) < \infty$ the dual value function of the robust problem satisfies*

$$v(y) = \inf_{Q\in\mathcal{Q}_e} \big(v_Q(y) + \gamma(Q)\big) = \inf_{Q\in\mathcal{Q}^f_e} \big(v_Q(y) + \gamma(Q)\big).$$

Proof: As for the proof of the first identity, suppose $Z_0 \in \mathcal{Z}\backslash\mathcal{Z}_e$ and $h_0 \in \mathcal{D}(y)$ are such that $\mathbb{E}[\, Z_0 V(h_0/Z_0)\,] < \infty$. Due to our assumption (9), we may choose $Z_1 \in \mathcal{Z}_e$ and $h_1 \in \mathcal{D}(y)$ such that $\mathbb{E}[\, Z_1 V(h_1/Z_1)\,] < \infty$. Now let $Z_t := tZ_1 + (1-t)Z_0 \in \mathcal{Z}_e$ and $h_t := th_1 + (1-t)h_0$ for $0 < t \leq 1$. Since the function $t \mapsto \mathbb{E}[\, Z_t V(h_t/Z_t)\,] + \gamma(Z_t)$ is convex and takes finite values, it is upper semicontinuous and we get $v_{Z_0}(y) + \gamma(Z_0) \geq \limsup_{t\downarrow 0}(v_{Z_t}(y) + \gamma(Z_t))$. This proves the first identity. The second identity follows from the fact that for $Q \sim \mathbb{P}$ we have $v_Q \equiv \infty$ as soon as $u_Q \equiv \infty$; see the proof of [35, Lemma 3.5]. □

Remark 4.5 In the sequel, we will sometimes use variants of the upper semicontinuity argument in the preceding proof. For a convex set $\mathcal{Z}' \subset \mathcal{Z}$ and $Z_0, Z_1 \in \mathcal{Z}'$ let $Z_t := tZ_1 + (1-t)Z_0$. If $f : \mathcal{Z}' \to \mathbb{R} \cup \{\infty\}$ is a convex functional and $f(Z_t) < \infty$ for $0 < t < 1$, then $t \mapsto f(Z_t)$ is upper semicontinuous on $[0, 1]$. If f is moreover lower semicontinuous (e.g., with respect to $\mathbb{P}$-a.s. convergence), then $t \mapsto f(Z_t)$ is even continuous on $[0, 1]$. Due to (4), this argument applies to $\mathcal{Z}' := \mathcal{Z}$ and $f(Z) := \gamma(Z)$. It also works for $\mathcal{Z}' := \mathcal{Z}^f$ and $f(Z) := u_Z(x)$; see [35, Lemma 3.3].

Lemma 4.6 *We have*

$$\begin{aligned} u(x) &= \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}} \big(E_Q[\, U(g)\,] + \gamma(Q)\big) = \inf_{Q\in\mathcal{Q}} \big(u_Q(x) + \gamma(Q)\big) \\ &= \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}_e} \big(E_Q[\, U(g)\,] + \gamma(Q)\big) = \inf_{Q\in\mathcal{Q}_e} \big(u_Q(x) + \gamma(Q)\big). \end{aligned}$$

Proof: Take $\varepsilon \in (0, 1)$ and let $c := 1 + u(x+1) - U(\varepsilon) \wedge 0$ so that

$$u(x+\varepsilon) \geq \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}} \big(E_Q[\, U(\varepsilon + g)\,] + \gamma(Q)\big) = \sup_{g\in\mathcal{C}(x)} \inf_{Z\in\mathcal{Z}(c)} \big(\mathbb{E}[\, ZU(\varepsilon + g)\,] + \gamma(Z)\big).$$

On the one hand, the function $U(\cdot + \varepsilon)$ is bounded from below, and so $Z \mapsto \mathbb{E}[\, ZU(\varepsilon + g)\,]$ is a weakly lower semicontinuous affine functional on $\mathcal{Z}(c)$. Furthermore, $Z \mapsto \gamma(Z)$ is also weakly lower semicontinuous, and the set $\mathcal{Z}(c)$ is convex and weakly compact by Lemma 4.1. On the other hand, for each $Z \in \mathcal{Z}(c)$, $g \mapsto \mathbb{E}[\, ZU(\varepsilon + g)\,]$ is a concave functional defined on the convex set $\mathcal{C}(x)$. Thus, the conditions of the lopsided minimax theorem [1, Chapter 6, p. 295] are satisfied, and so

$$\sup_{g\in\mathcal{C}(x)} \inf_{Z\in\mathcal{Z}(c)} \big(\mathbb{E}[\, ZU(\varepsilon + g)\,] + \gamma(Z)\big) = \inf_{Z\in\mathcal{Z}(c)} \sup_{g\in\mathcal{C}(x)} \big(\mathbb{E}[\, ZU(\varepsilon + g)\,] + \gamma(Z)\big).$$

Since this expression is bounded above by $u(x+\varepsilon) < c + U(\varepsilon) \wedge 0$, we may replace $\mathcal{Z}(c)$ by $\mathcal{Z}$. Hence, we arrive at

$$\begin{aligned} u(x+\varepsilon) &\geq \inf_{Z\in\mathcal{Z}} \sup_{g\in\mathcal{C}(x)} \big(\mathbb{E}[\,ZU(\varepsilon+g)\,] + \gamma(Z)\big) \geq \inf_{Z\in\mathcal{Z}} \sup_{g\in\mathcal{C}(x)} \big(\mathbb{E}[\,ZU(g)\,] + \gamma(Z)\big) \\ &\geq \sup_{g\in\mathcal{C}(x)} \inf_{Z\in\mathcal{Z}} \big(\mathbb{E}[\,ZU(g)\,] + \gamma(Q)\big) = u(x). \end{aligned}$$

Sending $\varepsilon \downarrow 0$ and using the continuity of u yields the first part of the lemma.

We still have to show that $\mathcal{Z}$ may be replaced by $\mathcal{Z}_e$. To this end, let $Z_0 \in \mathcal{Z}^f \backslash \mathcal{Z}_e$. By assumption (8) there also exists some $Z_1 \in \mathcal{Z}_e^f$. Remark 4.5 then gives $u_{Z_0}(x) = \lim_{t\downarrow 0} u_{Z_t}(x)$, where $Z_t := (1-t)Z_0 + tZ_1 \in \mathcal{Z}_e$ for $0 < t \leq 1$. Hence, using the first part of this proof,

$$\begin{aligned} u(x) = \inf_{Z\in\mathcal{Z}_e} \big(u_Z(x) + \gamma(Z)\big) &\geq \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}_e} \big(E_Q[\,U(g)\,] + \gamma(Q)\big) \\ &\geq \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}} \big(E_Q[\,U(g)\,] + \gamma(Q)\big) = u(x). \end{aligned}$$

□

Proof of Theorem 2.4: By Lemma 4.6, (7), and Lemma 4.4,

$$\begin{aligned} u(x) &= \inf_{Q\in\mathcal{Q}_e} \big(u_Q(x) + \gamma(Q)\big) = \inf_{Q\in\mathcal{Q}_e^f} \big(u_Q(x) + \gamma(Q)\big) \\ &= \inf_{Q\in\mathcal{Q}_e^f} \inf_{y>0} \big(v_Q(y) + \gamma(Q) + xy\big) = \inf_{y>0} \big(v(y) + xy\big), \end{aligned}$$

which is the first identity in (10).

To prove the second one, we first observe that v is convex due to the convexity of $(Z,h) \mapsto \mathbb{E}[\,ZV(h/Z)\,] + \gamma(Z)$. Next we will prove that v is lower semicontinuous on $[0,\infty)$ if we define $v(0) := V(0) := \lim_{y\downarrow 0} V(y)$. This will then imply that v is the conjugate function of u according to standard biduality results; see, e.g., [18, Proposition A.6 (b)]. To this end, take a sequence $y_n > 0$ converging to $y \geq 0$. There is nothing to show if $\liminf_n v(y_n) = \infty$, so we may assume that $\sup_n v(y_n) < \infty$. By Lemma 4.3 there are $\widehat{h}_n \in \mathcal{D}(y_n)$ and $\widehat{Z}_n \in \mathcal{Z}$ such that $v(y_n) = \mathbb{E}[\,\widehat{Z}_n V(\widehat{h}_n/\widehat{Z}_n)\,] + \gamma(\widehat{Z}_n)$. By (26) we have $v(y_n) \geq V(y_n) + \gamma(\widehat{Z}_n)$. Since γ is bounded from below, we must necessarily have $V(y) < \infty$. Moreover, all $\widehat{Z}_n$ must belong to some $\mathcal{Z}(c)$ for some finite constant c. As above, we can pass to a sequence of convex combinations, which converges $\mathbb{P}$-a.s. to some $(\widehat{h}, \widehat{Z})$. Using again (23) and Lemma 4.1 yields $(\widehat{h}, \widehat{Z}) \in \mathcal{D}(y) \times \mathcal{Z}(c)$, while convexity, lower semicontinuity of (27), and Lemma 4.1 give $v(y) \leq \mathbb{E}[\,\widehat{Z}V(\widehat{h}/\widehat{Z})\,] + \gamma(\widehat{Z}) \leq \liminf_n v(y_n)$.

The identities in (11) can be proved as in [25, Lemma 3.5]. □

Proof of Proposition 2.5: The strict convexity of v will imply the differentiability of u; see, e.g., [31, Theorem V.26.3]. So suppose by way of contradiction that $0 < y_0 < y_1$ are such that v is finite and affine on $[y_0, y_1]$. By Lemma 4.3 there are $Z_i \in \mathcal{Z}$ and $h_i \in \mathcal{D}(y_i)$

such that $v(y_i) = \mathbb{E}[\,Z_i V(h_i/Z_i)\,] + \gamma(Z_i)$. We let $\widetilde{y} := (y_1 + y_0)/2$, $\widetilde{h} := (h_1 + h_0)/2$ etc. Then $\widetilde{h} \in \mathcal{D}(\widetilde{y})$ due to (23). Hence, the affinity of v and (25) imply that

$$\begin{aligned} v(\widetilde{y}) = \frac{v(y_1) + v(y_0)}{2} &= \frac{1}{2}\Big(\mathbb{E}[\,Z_1 V(h_1/Z_1)\,] + \gamma(Z_1) + \mathbb{E}[\,Z_0 V(h_0/Z_0)\,] + \gamma(Z_0)\Big) \\ &\geq \mathbb{E}[\,\widetilde{Z} V(\widetilde{h}/\widetilde{Z})\,] + \gamma(\widetilde{Z}) \geq v(\widetilde{y}). \end{aligned}$$

Hence, the strict convexity of γ implies that $\mathbb{P}$-a.s. $Z_0 = Z_1$. But then we must also $h_1 = h_0$ $\mathbb{P}$-a.s. on $\{Z_i > 0\}$, due to the strict convexity (25). Thus, we get $v(y_0) = v(y_1)$. However, taking a strictly positive $h \in \mathcal{D}(1)$ (e.g. the density of some $P^* \in \mathcal{M}$) we have $\widetilde{h}_1 := h_0 + (y_1 - y_0)h \in \mathcal{D}(y_1)$ and $\widetilde{h}_1 > h_0$ so that

$$v(y_1) \leq \mathbb{E}[\,Z_0 V(\widetilde{h}_1/Z_0)\,] < \mathbb{E}[\,Z_0 V(h_0/Z_0)\,] = v(y_0) = v(y_1),$$

which is the desired contradiction. □

We turn now to the existence and characterization of optimal strategies.

Lemma 4.7 *Under condition* (12)*, for all $x > 0$ there exists some $\widehat{g} \in \mathcal{C}(x)$ such that $\inf_{Q\in\mathcal{Q}} \big(E_Q[\,U(\widehat{g})\,] + \gamma(Q)\big) = u(x)$.*

Proof: Due to our assumption (12) and [35, Lemma 3.5], we have $\mathcal{Q}_e^f = \mathcal{Q}_e$. In particular, we have $E_Q[\,U^+(g)\,] < \infty$ for all $Q \in \mathcal{Q}_e$ and $g \in \mathcal{C}(x)$ by (24), and so the expectations $E_Q[\,U(g)\,]$ are defined in the standard way. Moreover,

$$\frac{u_Q(x)}{x} \longrightarrow 0 \qquad \text{as } x \uparrow \infty$$

for each $Q \in \mathcal{Q}_e$; see [26, Note 1]. Hence it follows from the proof of [26, Eq. (25)] that the mapping $\mathcal{C}(x) \ni g \mapsto E_Q[\,U(g)\,]$ is upper semicontinuous with respect to $\mathbb{P}$-almost-sure convergence (note that the proof of Eq. (25) in [26] does not use the assumption that (g^n) is a maximizing sequence). Hence, $\mathcal{C}(x) \ni g \mapsto \inf_{Q\in\mathcal{Q}_e} \big(E_Q[\,U(g)\,] + \gamma(Q)\big)$ is also upper semicontinuous with respect to $\mathbb{P}$-almost-sure convergence. Now let $(\widetilde{g}_n)$ be a maximizing sequence in $\mathcal{C}(x)$. By the usual Komlos-type argument there is a sequence $g_n \in \operatorname{conv}\{\widetilde{g}_n, \widetilde{g}_{n+1}, \dots\}$ converging $\mathbb{P}$-a.s. to some $\widehat{g} \geq 0$. We have $\widehat{g} \in \mathcal{C}(x)$ due to (23). Moreover, the concavity of the functional $g \mapsto \inf_{Q\in\mathcal{Q}_e} \big(E_Q[\,U(g)\,] + \gamma(Q)\big)$ implies that (g_n) is again a maximizing sequence, while its upper semicontinuity yields that $\inf_{Q\in\mathcal{Q}_e} \big(E_Q[\,U(\widehat{g})\,] + \gamma(Q)\big) \geq u(x)$.

We note next that the set $\{Q \in \mathcal{Q} \,|\, E_Q[\,U^-(\widehat{g})\,] = \infty\}$ must be empty, for otherwise it would have a nonvoid intersection with $\mathcal{Q}_e$. Hence, for $Q \in \mathcal{Q}\backslash\mathcal{Q}_e$ and $Q_0 \in \mathcal{Q}_e$, $E_Q[\,U(\widehat{g})\,]$ is the limit as $t \uparrow 1$ of $E_{Q_t}[\,U(\widehat{g})\,]$, where $Q_t := tQ + (1-t)Q_0 \in \mathcal{Q}_e$. By Remark 4.5, we also have $\gamma(Q_t) \to \gamma(Q)$. This shows that we have $\inf_{Q\in\mathcal{Q}} \big(E_Q[\,U(\widehat{g})\,] + \gamma(Q)\big) \geq u(x)$. □

Proof of Theorem 2.6: The existence of an optimal strategy $\widehat{X}$ follows from Lemma 4.7. The assertion that $u'(\infty-)=0$ follows from the fact that $u(x)/x \to 0$ as $x \uparrow \infty$, which is itself a consequence of assumption (12) and [26, Note 1]. The second identity in (13) follows from the first and the duality relations between u and v.

Now let $y>0$ be such that $v(y)+xy=u(x)$. Such a y exists due to the fact that $v'(0+)=-\infty$ and $v'(\infty-)=0$. We take a solution $(\widehat{Q},\widehat{Y})$ to the dual problem at level y and denote by $\widehat{Z}$ the density process of $\widehat{Q}$ with respect to $\mathbb{P}$. By an abuse of notation, we will also write $\widehat{Z}_T=\widehat{Z}$. Our next goal is to show that $(\widehat{Q},\widehat{X})$ is a saddlepoint for the robust problem. To this end, take any $Z_1 \in \mathcal{Z}_e$ and let $Z_t := (1-t)\widehat{Z}+tZ_1 \in \mathcal{Z}_e$ for $0<t\le 1$.

We first claim that $v_{Z_t}(y)+\gamma(Z_t) \to v(y)$ as $t \downarrow 0$. To prove this claim, let $\widehat{h}, h_1 \in \mathcal{D}(y)$ be such that $v_{\widehat{Z}}(y)=\mathbb{E}[\,\widehat{Z}V(\widehat{h}/\widehat{Z})\,]$ and $v_{Z_1}(y)=\mathbb{E}[\,Z_1V(h_1/Z_1)\,]$, and let $h_t:=(1-t)\widehat{h}+th_1 \in \mathcal{D}(y)$. By the convexity of $(y,z)\mapsto zV(y/z)$ we have

$$\begin{aligned} v(y) \le v_{Z_t}(y)+\gamma(Z_t) &\le \mathbb{E}\Big[\,Z_tV\Big(\frac{h_t}{Z_t}\Big)\,\Big]+\gamma(Z_t)\\ &\le t\big(v_{Z_1}(y)+\gamma(Z_1)\big)+(1-t)\big(v_{\widehat{Z}}(y)+\gamma(\widehat{Z})\big), \end{aligned}$$

and our claim follows, since the right-hand side tends to $v(y)$ as $t\downarrow 0$.

Next, due to the duality relations (7) between v_{Z_t} and u_{Z_t}, we have $v_{Z_t}(y)+xy \ge u_{Z_t}(x)$. Moreover, as $t\downarrow 0$, $u_{Z_t}(x)+\gamma(Z_t)$ tends to $u_{\widehat{Z}}(x)+\gamma(\widehat{Z})$ according to Remark 4.5. Thus, we obtain

$$u(x)=v(y)+xy=\lim_{t\downarrow 0}(v_{Z_t}(y)+xy+\gamma(Z_t)) \ge \lim_{t\downarrow 0}(u_{Z_t}(x)+\gamma(Z_t))=u_{\widehat{Z}}(x)+\gamma(\widehat{Z}).$$

Thus, Lemma 4.6 implies that $u_{\widehat{Z}}(x)+\gamma(\widehat{Z})=u(x)$. Now we can conclude that

$$u(x)=u_{\widehat{Z}}(x)+\gamma(\widehat{Z}) \ge \mathbb{E}[\,\widehat{Z}U(\widehat{X}_T)\,]+\gamma(\widehat{Z}) \ge \inf_{Q\in\mathcal{Q}}\big(E_Q[\,U(\widehat{X}_T)\,]+\gamma(Q)\big)=u(x),$$

which finishes the proof that $(\widehat{Q},\widehat{X})$ is a saddlepoint.

Next, we show that $\widehat{X}_T$ coincides $\widehat{Q}$-a.s. with $I(\widehat{Y}_T)$. We have $0 \le V(\widehat{Y}_T)+\widehat{X}_T\widehat{Y}_T-U(\widehat{X}_T)$ $\widehat{Q}$-a.s. and

$$E_{\widehat{Q}}[\,V(\widehat{Y}_T)+\widehat{X}_T\widehat{Y}_T-U(\widehat{X}_T)\,]=v(y)+\mathbb{E}[\,\widehat{X}_T\widehat{Y}_T\widehat{Z}\,]-u(x)\le v(y)+xy-u(x)=0,$$

where we have used (23) and the fact that the process $\widehat{Y}\widehat{Z}$ belongs to $\mathcal{Y}(y)$ due to Lemma 4.2. Thus, $0=V(\widehat{Y}_T)+\widehat{X}_T\widehat{Y}_T-U(\widehat{X}_T)$ and in turn $\widehat{X}_T=I(\widehat{Y}_T)$ $\widehat{Q}$-a.s. We also get $\mathbb{E}[\,\widehat{X}_T\widehat{Y}_T\widehat{Z}\,]=xy$, and this implies that the process $\widehat{X}\widehat{Y}\widehat{Z}$ is a $\mathbb{P}$-martingale.

We will show next that u is strictly concave. The continuous differentiability of v will then follow by general principles (e.g., [31, Theorem V.26.3]) and from the duality relations (10). Suppose by way of contradiction that u is not strictly concave. Since u is strictly increasing with $u'(0+)=\infty$ and $u'(\infty-)=0$, there will be $0<x_0<x_1$ and $y>0$ such that $v(y)+x_iy=u(x_i)$ for $i=0,1$. Let $\widehat{X}^i \in \mathcal{X}(x_i)$ be the corresponding optimal solutions, and let $(\widehat{Q},\widehat{Y})$ be a solution to the dual problem at level y. Then we have both $\widehat{X}^0_T=I(\widehat{Y}_T)=\widehat{X}^1_T$ $\widehat{Q}$-a.s. and $E_{\widehat{Q}}[\,\widehat{X}^0_T\widehat{Y}_T\,]=x_0y<x_1y=E_{\widehat{Q}}[\,\widehat{X}^1_T\widehat{Y}_T\,]$, which is impossible. □

Proof of Corollary 2.7: The existence of a superhedging strategy for H with initial capital x follows from Theorem 2.6. That is, we have $H \in \mathcal{C}(x)$. Moreover, we have $\widehat{Y}_T \widehat{Z} \in \mathcal{D}(y)$ by Lemma 4.2, and hence

$$\sup_{h \in \mathcal{D}(y)} \mathbb{E}[\, Hh \,] \geq \mathbb{E}[\, H \widehat{Y}_T \widehat{Z} \,] = xy,$$

where the equality on the right follows from Theorem 2.6. Hence, due to (23), H cannot belong to any set $\mathcal{C}(x')$ with $x' < x$. $\square$

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