

Moderate deviations and functional LIL for super-Brownian motion

Alexander Schied

*Institut für Mathematik, Humboldt-Universität,
Unter den Linden 6, D-10099 Berlin, Germany*

Abstract: A moderate deviation principle and a Strassen type law of the iterated logarithm for the small-time propagation of super-Brownian motion are derived. Moderate deviation estimates which are uniform with respect to the starting point are developed in order to prove the law of the iterated logarithm. Our method also yields a functional central limit theorem.

1. Introduction

Let $M^+(\mathbb{R}^d)$ denote the space of positive finite measures on \mathbb{R}^d and endow it with the usual weak topology. Then super-Brownian motion X is an $M^+(\mathbb{R}^d)$ -valued diffusion. The law \mathbb{P}_μ of this process starting from $\mu \in M^+(\mathbb{R}^d)$ can be characterized as usual by the Laplace functionals of its transition probabilities:

$$(1.1) \quad \mathbb{E}_\mu \left[\exp(-\langle f, X_t \rangle) \right] = \exp(-\langle u(t), \mu \rangle),$$

where $t \geq 0$, f is a non-negative function in $C_b(\mathbb{R}^d)$, the space of bounded and continuous functions on \mathbb{R}^d , $\langle g, \nu \rangle$ denotes the integral of a function g with respect to some measure ν , and u is the unique positive mild solution of the reaction-diffusion equation

$$(1.2) \quad \begin{aligned} \frac{\partial}{\partial t} u(t, x) &= \frac{1}{2} \Delta u(t, x) - u(t, x)^2 \\ u(0, x) &= f(x). \end{aligned}$$

The goal of this paper is to derive a moderate deviation principle and a local law of the iterated logarithm of Strassen type for super-Brownian motion. Therefore it is necessary

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to center the process. It follows from (1.1) and (1.2) that $\mathbb{E}_\mu[\langle f, X_t \rangle] = \langle f, \mu P_t \rangle$, where $P_t(x, dy) = (2\pi t)^{d/2} \exp(-|x - y|^2/2t) dy$ is the Brownian transition semigroup. The centered process is thus given by

$$(1.3) \quad \widehat{X}_t = X_t - X_0 P_t \quad (t \geq 0).$$

It takes values in the set $M(\mathbb{R}^d)$ of finite signed measures. This space will be endowed with the coarsest topology such that the mappings $\mu \mapsto \langle f, \mu \rangle$ are continuous for each $f \in BL(\mathbb{R}^d)$, the space of bounded Lipschitz functions on \mathbb{R}^d . Note that, in contrast to the convex cone $M^+(\mathbb{R}^d)$, the topological space $M(\mathbb{R}^d)$ is *not* metrizable; see Dudley (1966). The set $C([0, 1]; M(\mathbb{R}^d))$ of all continuous $M(\mathbb{R}^d)$ -valued paths will be endowed with the compact open topology. When $M(\mathbb{R}^d)$ is equipped with the variation norm, it becomes a (non-separable) Banach space, and one can define Bochner integrals. For $\mu \in M^+(\mathbb{R}^d)$, we define \mathcal{H}_μ to be the set of all $\omega \in C([0, 1]; M(\mathbb{R}^d))$ of the form

$$\omega(t) = \int_0^t \dot{\omega}(s) ds \quad (t \geq 0),$$

for some Bochner integrable mapping $\dot{\omega} : [0, 1] \rightarrow M(\mathbb{R}^d)$ with $\dot{\omega}(t) \ll \mu$, for almost every t . Then we define the Gaussian rate function

$$(1.4) \quad I_\mu(\omega) = \begin{cases} \frac{1}{4} \int_0^1 \left\| \frac{d\dot{\omega}(t)}{d\mu} \right\|_{L^2(\mu)}^2 dt & \text{if } \omega \in \mathcal{H}_\mu, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 1.1: *Suppose $\beta : (0, 1] \rightarrow (0, 1]$ is a function such that $\beta(\alpha) \searrow 0$ as $\alpha \downarrow 0$. Then the distributions of the processes $\widehat{X}_t^{\alpha, \beta(\alpha)} := \beta(\alpha)^{-1} \widehat{X}_{\alpha\beta(\alpha)^2 t}$ ($0 \leq t \leq 1$) satisfy a large deviation principle on $C([0, 1]; M(\mathbb{R}^d))$ with scale α and good rate function I_μ . I.e., if $U \subset C([0, 1]; M(\mathbb{R}^d))$ is open, then*

$$\liminf_{\alpha \downarrow 0} \alpha \log \mathbb{P}_\mu [\widehat{X}^{\alpha, \beta(\alpha)} \in U] \geq - \inf_{\omega \in U} I_\mu(\omega);$$

if $A \subset C([0, 1]; M(\mathbb{R}^d))$ is closed, then

$$\limsup_{\alpha \downarrow 0} \alpha \log \mathbb{P}_\mu [\widehat{X}^{\alpha, \beta(\alpha)} \in A] \leq - \inf_{\omega \in A} I_\mu(\omega);$$

the level sets $\{I_\mu \leq c\}$ are compact, for each $c \geq 0$.

Due to the condition ‘ $\beta(\alpha) \searrow 0$ as $\alpha \downarrow 0$ ’, Theorem 1.1 describes the *moderate* deviations from the ‘law of large numbers’ $X_t \rightarrow \mu$ as $t \downarrow 0$ \mathbb{P}_μ -almost surely. The corresponding large deviations, i.e. the case $\beta \equiv 1$, are treated as Corollary 3 of Schied (1996). In this case one gets a non-Gaussian rate function, namely the energy with respect to the Kakutani-Hellinger metric. Note that in our Theorem 1.1 the state space $M(\mathbb{R}^d)$ is topologized in such a way that $\nu \mapsto \nu(\mathbb{R}^d)$ is a continuous mapping, whereas in the case $\beta \equiv 1$ only integration with respect to a continuous function f can only be continuous if f satisfies the decay condition $\sup_x |f(x)|(1 + |x|^p) < \infty$, for some fixed $p > d$; see Schied (1996). Therefore our moderate deviation principle does not extend automatically to super-Brownian motion with infinite initial measure μ . To handle the latter case one can topologize the space of all signed Radon measures by testing with Lipschitz functions having compact support in \mathbb{R}^d . Then slight modifications of our proofs below show that all main results of our paper have counterparts in this setting. Other results on sample path large deviations for super-Brownian motion can be found, for instance, in Deuschel and Wang (1993), Fleischmann et al. (1996), Schied (1996), and, for Le Gall’s Brownian Snake process, in Serlet (1997).

Now let \mathcal{A} denote the set of all closed subsets of $C([0, 1]; M(\mathbb{R}^d))$. The *Hausdorff topology* on \mathcal{A} is generated by the semi-metrics

$$\rho_j(A, B) := \sup_{x \in A} d_j(x, B) \vee \sup_{y \in B} d_j(y, A) \quad (j \in J; A, B \in \mathcal{A}),$$

if $\{d_j \mid j \in J\}$ is a filtering family of semi-distances on $C([0, 1]; M(\mathbb{R}^d))$; see Castaing and Valadier (1977), Chapter II, § 2. For $1/e$, define an \mathcal{A} -valued random variable Δ_δ as closure of the random set

$$\left\{ \left(\frac{\widehat{X}_{\varepsilon t}}{\sqrt{2\varepsilon \log \log \varepsilon^{-1}}} \right)_{0 \leq t \leq 1} \mid 0 < \varepsilon \leq \delta \right\}.$$

Then our law of the iterated logarithm reads as follows.

Theorem 1.2: *If $\mathcal{U} \subset \mathcal{A}$ is open and $K_\mu := \{I_\mu \leq 1/2\} \in \mathcal{U}$, then $\Delta_\delta \in \mathcal{U}$, for small δ , \mathbb{P}_μ -almost surely. In particular, Δ_δ converges in \mathbb{P}_μ -probability to K_μ as $\delta \downarrow 0$.*

Note that we cannot conclude almost sure convergence of Δ_δ to K_μ , because the first countability axiom fails for $M(\mathbb{R}^d)$ and hence for \mathcal{A} . But one could consider a weaker topology on $M(\mathbb{R}^d)$ generated by a countable separating subset of $BL(\mathbb{R}^d)$ instead of the topology induced by the full space $BL(\mathbb{R}^d)$. Then the Hausdorff topology on \mathcal{A} would be metrizable. There is a vast literature on functional laws of the iterated logarithm. The original statement is due to Strassen (1964). Here we will adopt Stroock’s idea of

using large deviations to get the key estimates (see e.g. Deuschel and Stroock (1989)). For ordinary Brownian motion, the *local* law has been stated in Gantert (1993). See also Mueller (1981). In the measure-valued setting, moderate deviations have been used in Wu (1994) and Dembo and Zajic (1995) to derive Strassen type laws for empirical measures and processes. However, our situation differs somewhat from the above, because super-Brownian motion does not have independent increments, and in addition, the rate function depends heavily on the starting point of the process. As a consequence Theorem 1.1 alone is not sufficient to provide the desired estimates.

So far we have discussed the cases where $\beta(\alpha) \searrow 0$ as $\alpha \downarrow 0$, and where $\beta \equiv 1$. If $\alpha \equiv 1$ we are in the regime of the following functional central limit theorem, which will be an immediate consequence of the proof of Theorem 1.1.

Corollary 1.3: *Suppose that $f_1, \dots, f_n \in BL(\mathbb{R}^d)$. Then, as $\beta \downarrow 0$, the laws with respect to \mathbb{P}_μ of the \mathbb{R}^n -valued processes*

$$\frac{1}{\beta}(\langle f_1, \widehat{X}_{\beta^2 t} \rangle, \dots, \langle f_n, \widehat{X}_{\beta^2 t} \rangle) \quad (0 \leq t \leq 1)$$

converge weakly on $C([0, 1]; \mathbb{R}^n)$ to a n -dimensional Brownian motion B with covariance $E[B_t^i B_t^j] = 2t \int f_i f_j d\mu$.

The paper is organized as follows. In Section 2 we will give exponential estimates for the Hölder norm of super-Brownian motion tested with some $f \in BL(\mathbb{R}^d)$, and these estimates will be uniform with respect to the starting point. They rely on an embedding theorem between certain Orlicz-Hölder spaces. It can be found in the final Section 7 of this paper. In Section 3 we will show (uniform) convergence of the multivariate Laplace functionals as required for an application of the Gärtner-Ellis theorem. Here we will also indicate a proof of the CLT of Corollary 1.3. The proof of Theorem 1.1 can be found in Section 4. In Section 5 we will state large deviation estimates that are uniform with respect to some parameter, even though the rate function depends on it, too. This generalizes the concept of ‘uniform large deviations’. The LIL of Theorem 1.2 will be proved in Section 6.

2. Uniform Hölder norm estimates for super-Brownian motion

In this section we apply the results of the Section 7 to super-Brownian motion. But let us first recall an estimate for its Laplace functionals, that will be used several times in this paper. Define, for $t \geq 0$ and $x \in \mathbb{R}$,

$$(2.1) \quad V_t x = \begin{cases} \frac{x}{1-tx} & \text{if } x < 1/t, \\ \infty & \text{otherwise.} \end{cases}$$

It has been shown in Theorem 9 of Schied (1996) that, for any bounded and measurable function f on \mathbb{R}^d ,

$$(2.2) \quad \langle V_t P_t f, \mu \rangle \leq \log \mathbb{E}_\mu [\exp \langle f, X_t \rangle] \leq \langle P_t V_t f, \mu \rangle.$$

These bounds will mostly be applied together with the Markov property of X and the following extension of the branching property (1.1).

$$(2.3) \quad \mathbb{E}_\mu [\exp \langle f, X_t \rangle] = \exp \left(\int \log \mathbb{E}_{\delta_x} [\exp \langle f, X_t \rangle] \mu(dx) \right),$$

which holds for $\mu \in M^+(\mathbb{R}^d)$, $t \geq 0$ and bounded measurable f taking arbitrary signs. Cf. Schied (1996), Lemma 6. For $g \in BL(\mathbb{R}^d)$, define the norm $\|g\|_{BL}$ by

$$\|g\|_{BL} = \|g\| + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|},$$

where $\|\cdot\|$ denotes the usual sup-norm used also for non-continuous functions f defined everywhere on \mathbb{R}^d , i.e., $\|f\| = \sup_x |f(x)|$. If w is a path in $C[0, 1]$ we will denote with $|w|_\gamma$ its Hölder norm with exponent $\gamma \in (0, 1]$:

$$(2.4) \quad |w|_\gamma = \sup \left\{ \frac{|w(t) - w(s)|}{|t - s|^\gamma} \mid s, t \in [0, 1], s \neq t \right\}.$$

Lemma 2.1: *Suppose we are given three positive constants L , M and B and some $\gamma \in (0, 1/2)$. Then there exist $\beta_0 = \beta_0(M, B, d) > 0$ and $R = R(L, M, B, \gamma, d) > 0$ such that*

$$\mathbb{P}_\mu \left[|\langle f, \widehat{X}^{\alpha, \beta} \rangle|_\gamma \geq R \right] \leq e^{-L/\alpha},$$

whenever $\langle 1, \mu \rangle \leq M$, $\|f\|_{BL} \leq B$, $\beta \in (0, \beta_0)$ and $\alpha \in (0, 1]$.

Proof: Define

$$\psi(r, t) = \frac{tr^2}{1 - tr} + r\sqrt{td} \quad (t, r \geq 0, t \cdot r < 1).$$

Then choose β_0 such that $\psi(B\beta, \beta^2) < \beta^{-2}$, for all $\beta \in (0, \beta_0)$. When using in addition the following simple estimate

$$(2.5) \quad \|P_t g - P_s g\| \leq \|P_{t-s} g - g\| \leq \|g\|_{BL} \sqrt{d|t - s|},$$

for $g \in BL(\mathbb{R}^d)$, one can prove as in Lemma 13 of Schied (1996) that, for f and β as above and $s, t \in [0, 1]$, $s \neq t$,

$$(2.6) \quad \mathbb{E}_\mu \left[\exp \left(\frac{1}{\alpha \sqrt{|t-s|}} |\langle f, \widehat{X}_t^{\alpha, \beta} \rangle - \langle f, \widehat{X}_s^{\alpha, \beta} \rangle| \right) \right] \\ \leq 2 \cdot \exp \left(M \left(B \sqrt{d/\alpha} + V_{\alpha\beta^2} \psi(B/\alpha\beta \sqrt{|t-s|}, \alpha\beta^2 |t-s|) \right) \right) \\ = 2 \cdot \exp \left(\frac{M}{\alpha} \left(B \sqrt{\alpha d} + V_{\beta^2} \left[\frac{B^2}{1 - \beta \sqrt{|t-s|}} + B \sqrt{\alpha d} \right] \right) \right).$$

From here we conclude that there is a constant $\kappa > 0$ depending only on M , B and d such that (2.6) is bounded above by $\kappa^{1/\alpha}$, whenever $\beta < \beta_0$. Hence the lemma follows from Corollary 7.1 below. \square

3. Uniform convergence of the Laplace functionals

In this section we prove convergence of the multivariate Laplace functionals of the process $\widehat{X}^{\alpha, \beta(\alpha)}$, as needed for the application of the Gärtner-Ellis theorem. Moreover, we will show that the rate of convergence is locally uniform with respect to the starting point μ . This will enable us to derive uniform large deviation estimates in Section 5.

For $0 \leq t_1 < \dots < t_n \leq 1$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ define $Z_{t_1 \dots t_n}(\lambda_1, \dots, \lambda_n)$ by

$$Z_{t_1 \dots t_n}(\lambda_1, \dots, \lambda_n) = E \left[\exp \left(\sum_{i=1}^n \lambda_i B_{2t_i} \right) \right],$$

where B is a standard one-dimensional Brownian motion starting from 0. Then the following recursion formula holds for $n \geq 2$.

$$(3.1) \quad Z_{t_1 \dots t_n}(\lambda_1, \dots, \lambda_n) = Z_{t_1 \dots t_{n-1}}(\lambda_1, \dots, \lambda_{n-2}, \lambda_{n-1} + \lambda_n) \cdot Z_{t_n - t_{n-1}}(\lambda_n).$$

If $\mu \in M^+(\mathbb{R}^d)$ and $f_1, \dots, f_n \in C_b(\mathbb{R}^d)$ let

$$(3.2) \quad \log Z_{\mu, t_1 \dots t_n}(f_1, \dots, f_n) := \int \log Z_{t_1 \dots t_n}(f_1(x), \dots, f_n(x)) \mu(dx).$$

The next Lemma will only be applied in the case where the functions f_i^β below are all identical to f_i , for all i . However, the more general statement is necessary, because of an induction argument in the proof.

Lemma 3.1: *Suppose we are given positive constants $K, \beta_0, b_1, \dots, b_n$ and $0 \leq t_1 < \dots < t_n \leq 1$. Assume furthermore that $f_i \in BL(\mathbb{R}^d)$ with $\|f_i\|_{BL} \leq K$ and that f_i^β is a bounded*

and measurable function on \mathbb{R}^d with $\|f_i - f_i^\beta\| \leq b_i\beta$ for $i = 1, \dots, n$ and $0 < \beta < \beta_0$. Then there are two constants C and β_1 depending only on K, β_0, n, d and b_1, \dots, b_n such that

$$\left| \alpha \log \mathbb{E}_\mu \left[\exp \left(\frac{1}{\alpha} \sum_{i=1}^n \langle f_i^\beta, \widehat{X}_{\alpha\beta^2 t_i} \rangle \right) \right] - \log Z_{\mu, t_1 \dots t_n}(f_1, \dots, f_n) \right| \leq \langle 1, \mu \rangle C \beta,$$

whenever $\mu \in M^+(\mathbb{R}^d)$, $0 < \alpha \leq 1$ and $0 < \beta < \beta_1$.

Proof: Without loss of generality we may assume $t_1 = 0$. Then the assertion is trivial if $n = 1$. To prove the general case we proceed by induction. Applying the Markov property, (2.2), and (2.3) yields

$$\begin{aligned} \alpha \log \mathbb{E}_{\delta_x} \left[\exp \left(\frac{1}{\alpha} \sum_{i=1}^n \langle f_i^\beta, \widehat{X}_{\alpha\beta^2 t_i} \rangle \right) \right] &= \\ &= \alpha \log \mathbb{E}_{\delta_x} \left[\exp \left(\frac{1}{\alpha} \sum_{i=1}^{n-1} \langle \widetilde{f}_i^\beta, \widehat{X}_{\alpha\beta^2 t_i} \rangle \right) \right] + P_{\alpha\beta^2 t_{n-1}} g^\beta(x) - \frac{1}{\beta} P_{\alpha\beta^2 t_n} f_n^\beta(x), \end{aligned}$$

where $g^\beta(y) = \alpha \log \mathbb{E}_{\delta_x} \left[\exp \left(\frac{1}{\alpha} \langle f_n^\beta, \widehat{X}_{\alpha\beta^2 r} \rangle \right) \right]$, $r = t_n - t_{n-1}$, $\widetilde{f}_{n-1}^\beta = f_{n-1}^\beta + \beta g^\beta$ and $\widetilde{f}_i^\beta = f_i^\beta$, for $i \leq n-2$. Let us investigate first the asymptotics of g^β . To this end suppose that $\beta < \beta_1^{(n)} := 1 \wedge \beta_0 \wedge (2K + 2b_n)^{-1}$. Then the upper bound of (2.2) gives us that

$$\begin{aligned} R_0^\beta &:= g^\beta - \frac{1}{\beta} P_{\alpha\beta^2 r} f_n^\beta \leq \frac{1}{\beta} P_{\alpha\beta^2 r} \left(V_{\beta r} f_n^\beta - f_n^\beta \right) \\ &= \frac{1}{\beta^2 r} \sum_{k=2}^{\infty} P_{\alpha\beta^2 r} (\beta r f_n^\beta)^k \leq r P_{\alpha\beta^2 r} (f_n^\beta)^2 + 2(K + b_n)^3 \beta. \end{aligned}$$

Analogously, using the lower bound of (2.2), $R_0^\beta \geq r(P_{\alpha\beta^2 r} f_n^\beta)^2 - 2(K + b_n)^3 \beta$. But using (2.5) and $\|f^2\|_{BL} \leq 2\|f\| \cdot \|f\|_{BL}$ one can show that $\|r P_{\alpha\beta^2 r} (f_n^\beta)^2 - r(f_n)^\beta\|$ and $\|r(P_{\alpha\beta^2 r} f_n^\beta)^2 - r(f_n)^\beta\|$ are both less than $(2K + b_n)b_n\beta + 2\|f\| \cdot \|f\|_{BL} (\alpha\beta^2 r d)^{1/2}$. Hence there is a constant \hat{b} such that $\|R_0^\beta - r(f_n)^\beta\| \leq \hat{b} \cdot \beta$ for all $\beta < \beta_1^{(n)}$.

Now we conclude for $R^\beta := P_{\alpha\beta^2 t_{n-1}} g^\beta - \frac{1}{\beta} P_{\alpha\beta^2 t_n} f_n^\beta = P_{\alpha\beta^2 t_{n-1}} R_0^\beta$ that

$$\begin{aligned} \|R^\beta - r f_n^2\| &\leq \|R_0^\beta - r f_n^2\| + \|P_{\alpha\beta^2 r} (f_n^\beta)^2 - (f_n)^\beta\| \\ &\leq \hat{b} \cdot \beta + 2\|f\| \cdot \|f\|_{BL} \sqrt{d\alpha\beta^2 t_{n-1}}. \end{aligned}$$

Hence there exists a constant c' depending only on K, b_n and d such that $\|R^\beta - r f_n^2\| \leq c' \beta$ for all $\beta < \beta_1^{(n)}$. Thus our assertion will be proved by induction once we have shown that

the functions \tilde{f}_{n-1}^β and $\tilde{f}_{n-1} := f_{n-1} + f_n$ satisfy again the assumptions of our lemma. But, for small β ,

$$\begin{aligned} \|\tilde{f}_{n-1}^\beta - \tilde{f}_{n-1}\| &\leq \|f_{n-1}^\beta - f_{n-1}\| + \|\beta g^\beta - f_n\| \leq b_n \beta + \beta \|R_0^\beta\| + \|P_{\alpha\beta^2 r} f_n^\beta - f_n\| \\ &\leq \tilde{b}_{n-1} \beta, \end{aligned}$$

for a suitable constant \tilde{b}_{n-1} . Moreover, $\tilde{f}_{n-1} \in BL(\mathbb{R}^d)$ with $\|\tilde{f}_{n-1}\|_{BL} \leq 2K$, and the lemma is proved. \square

Proof of Corollary 1.3: Consider the particular case $\alpha \equiv 1$ in Lemma 2.1. The Arzela-Ascoli theorem then shows that under \mathbb{P}_μ the distributions of the \mathbb{R}^n -valued stochastic processes $(\langle f_1, \widehat{X}^{1,\beta} \rangle, \dots, \langle f_n, \widehat{X}^{1,\beta} \rangle) / \beta$ are tight on $C([0, 1]; \mathbb{R}^n)$, as $\beta \downarrow 0$. In addition Lemma 3.1 implies that the corresponding finite dimensional marginal distributions converge weakly to those of an n -dimensional Wiener process W with covariance $E[W_t^i W_t^j] = 2t \int f_i f_j d\mu$. \square

4. Proof of Theorem 1.1

Let us first recall a basic fact about the topological space $\Omega := C([0, 1]; M(\mathbb{R}^d))$. It follows immediately from Proposition 1.6 i) and Theorem 1.7 of Jakubowski (1986).

Lemma 4.1: *Suppose $\omega_0 \in \Omega$. Then the sets of the form*

$$\bigcap_{i=1}^k \left\{ \omega \in \Omega \mid |\langle g_i, \omega(t) \rangle - \langle g_i, \omega_0(t) \rangle| < \delta, \ 0 \leq t \leq 1 \right\},$$

with $k \in \mathbb{N}$, $g_1, \dots, g_k \in BL(\mathbb{R}^d)$ and $\delta > 0$, are a base for the neighborhood system of ω_0 in Ω .

Fix $n \in \mathbb{N}$, $0 < t_1 < \dots < t_n \leq 1$ and $g_1, \dots, g_k \in BL(\mathbb{R}^d)$. Denote by H the linear hull of g_1, \dots, g_k and by H_n the n -fold direct sum $H \oplus \dots \oplus H$. H' and H'_n will be the corresponding dual spaces. All these vector spaces are of course finite dimensional and thus carry a unique locally convex topology. There is a natural projection $p_{H'_n} : (M(\mathbb{R}^d))^n \rightarrow H'_n$ given by

$$(4.1) \quad p_{H'_n}(\vec{\mu})(\vec{f}) = \sum_{i=1}^n \langle f_i, \mu_i \rangle,$$

for $\vec{\mu} = (\mu_1, \dots, \mu_n) \in (M(\mathbb{R}^d))^n$ and $\vec{f} = (f_1, \dots, f_n) \in H_n$. Now suppose that we are given a function $\beta(\alpha)$ such that $\beta(\alpha) \searrow 0$ as $\alpha \downarrow 0$. Then, for $\mu \in M^+(\mathbb{R}^d)$, we define Q_μ^α to be the law of $p_{H'_n}((\widehat{X}_{t_1}^{\alpha, \beta(\alpha)}, \dots, \widehat{X}_{t_n}^{\alpha, \beta(\alpha)}))$ under \mathbb{P}_μ .

Lemma 3.1 implies that

$$\alpha \log \int \exp\left(\frac{1}{\alpha} u(\vec{f})\right) Q_\mu^\alpha(du) \longrightarrow \log Z_{\mu, t_1 \dots t_n}(\vec{f}) \quad \text{for all } \vec{f} \in H_n,$$

as $\alpha \downarrow 0$. Note that the functional $\log Z_{\mu, t_1 \dots t_n}$ is differentiable throughout H_n and finite everywhere. Hence the Gärtner-Ellis theorem (cf. Dembo and Zeitouni (1993), p. 45) implies that the measures Q_μ^α satisfy a large deviation principle with scale α and good rate function

$$I_\mu^{H, t_1 \dots t_n}(u) = \sup_{\vec{f} \in H_n} \left(u(\vec{f}) - \log Z_{\mu, t_1 \dots t_n}(\vec{f}) \right) \quad (u \in H'_n).$$

Now we claim that, as $\alpha \downarrow 0$, the H' -valued processes $(p_{H'}(\widehat{X}_t^{\alpha, \beta(\alpha)}))_{0 \leq t \leq 1}$ satisfy a large deviation principle in $C([0, 1]; H')$ with good rate function

$$(4.2) \quad I_\mu^H(\eta) = \sup_{0 \leq t_1 < \dots < t_n \leq 1} I_\mu^{H, t_1 \dots t_n}(\eta(t_1), \dots, \eta(t_n)) \quad (\eta \in C([0, 1]; H')).$$

Indeed if $\alpha_n \rightarrow 0$, then Lemma 2.1 and the Arzela-Ascoli theorem show that the laws of $p_{H'}(\widehat{X}^{\alpha_n, \beta(\alpha_n)})$ are exponentially tight on $C([0, 1]; H')$ with scale (α_n) . Now apply Lemma 19 of Schied (1996) with $E := C([0, 1]; H')$ and J denoting the set of all finite ordered subsets of $[0, 1]$.

It is easy to see that the spaces $C([0, 1]; H')$ form a projective system indexed by the vector spaces H as above (cf. Section 4.6 of Dembo and Zeitouni (1993)). Its projective limit $\varprojlim C([0, 1]; H')$ will be denoted by \mathcal{X} . It follows from Lemma 4.1 that $C([0, 1]; M(\mathbb{R}^d))$ is homeomorphic to a subset of \mathcal{X} , which will be identified with Ω in the sequel. Hence \mathbb{P}_μ extends to \mathcal{X} , and the Dawson-Gärtner theorem (cf. Dembo and Zeitouni (1993), p. 144) asserts that the processes $\widehat{X}^{\alpha, \beta(\alpha)}$ satisfy a large deviation principle on \mathcal{X} with good rate function

$$(4.3) \quad \widetilde{I}_\mu(\omega) = \sup_H I_\mu^H(q_H(\omega)) \quad (\omega \in \mathcal{X}),$$

where q_H denotes the canonical projection from \mathcal{X} to $C([0, 1]; H')$.

Using induction one proves that, for $t_0 := 0$, $u = (u^1, \dots, u^n) \in H'_n$ and $u^0 := 0$,

$$(4.4) \quad I_\mu^{H, t_1 \dots t_n}(u) = \sum_{i=1}^n (t_i - t_{i-1}) \cdot I_\mu^{H, 1} \left(\frac{u^i - u^{i-1}}{t_i - t_{i-1}} \right).$$

Hence it follows from the proof of Theorem 1 (b) of Dembo and Zajic (1995) that

$$(4.5) \quad \widetilde{I}_\mu(\omega) = \begin{cases} I_\mu(\omega) & \text{if } \omega \in \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

where I_μ is as in (1.4). Therefore we can restrict the large deviation principle in \mathcal{X} to $\Omega = C([0, 1]; M(\mathbb{R}^d))$ by Lemma 4.1.5 (b) of Dembo and Zeitouni (1993), and Theorem 1.1 is proved. \square

5. Locally uniform large deviation estimates

In this section our goal is to prove a version of the lower bound in Theorem 1.1 which holds locally uniform with respect to the starting point. Our result is Proposition 5.4 below. Fix $0 < t_1 < \dots < t_n \leq 1$, and $g_1, \dots, g_k \in BL(\mathbb{R}^d)$, and assume g_1, \dots, g_k to be linearly independent. The vector spaces H , H' , H_n and H'_n will be defined as in the previous section, and $\log Z_\mu$ will be the restriction to H_n of the functional $\log Z_{\mu, t_1, \dots, t_n}$ defined in (3.2). The Legendre transform of $\log Z_\mu$ will be denoted by J_μ .

Lemma 5.1: *Fix $u \in H'_n$ and let D denote the set of all $\mu \in M^+(\mathbb{R}^d)$ for which $J_\mu(u) < \infty$. Then there exist vector spaces $\tilde{H}^1, \dots, \tilde{H}^n \subset H$ such that, for each $\mu \in D$, there is a unique $\vec{f}_\mu \in \tilde{H}^1 \oplus \dots \oplus \tilde{H}^n$ satisfying $J_\mu(u) = u(\vec{f}_\mu) - \log Z_\mu(\vec{f}_\mu)$, and $J_\mu(w) > w(\vec{f}_\mu) - \log Z_\mu(\vec{f}_\mu)$, for each $w \in H'_n$ which is different from u . Furthermore, if $g_1^i, \dots, g_{k_i}^i$ is a basis of \tilde{H}^i , then \vec{f}_μ regarded as a functional of $\mu \in D$ can be expressed as continuous function of the coefficients $\int g_l^i g_m^i d\mu$.*

Proof: Suppose first that $n = 1$. Write H as $H = H^0 \oplus \tilde{H}^1$, where $H^0 := \{g \in H \mid u(g) = 0\}$. Then, for each $\mu \in D$, $(f, g) \mapsto \int fg d\mu =: (f, g)_\mu$ defines a positive definite bilinear form on \tilde{H}^1 , because otherwise there would exist a function $g \in \tilde{H}^1$ with $u(g) > 0$ and $\log Z_\mu(g) = t_1(g, g)_\mu = 0$, which would imply that $J_\mu(u) = \infty$. Hence there is a unique $f_\mu \in \tilde{H}^1$ such that

$$(5.1) \quad u(f) = 2t_1(f_\mu, f)_\mu \quad (f \in \tilde{H}^1).$$

Then f_μ satisfies $J_\mu(u) = u(f_\mu) - \log Z_\mu(f_\mu) = t_1(f_\mu, f_\mu)_\mu$, and f_μ depends continuously on $\int g_l^1 g_m^1 d\mu$ ($l, m = 1, \dots, k_1$), if $g_1^1, \dots, g_{k_1}^1$ is a basis of \tilde{H}^1 . Now fix $\mu \in D$. To see that $J_\mu(w) > w(f_\mu) - \log Z_\mu(f_\mu)$ for all $w \in H$, $w \neq u$, decompose H^0 into $H_\mu^0 \oplus G_\mu^0$, where $H_\mu^0 = \{g \in H^0 \mid (f, g)_\mu = 0, \forall f \in H\}$ and G_μ^0 is orthogonal to \tilde{H}^1 with respect to $(\cdot, \cdot)_\mu$. Then, if $w(g) \neq 0$ for some $g \in H_\mu^0$, we have that $J_\mu(w) = \infty$, and the assertion is trivial. Otherwise w can be regarded as an element in the dual space of $\tilde{H}^1 \oplus G_\mu^0$. Since (5.1) extends to $f \in \tilde{H}^1 \oplus G_\mu^0$, the assertion follows from Lemma 2.3.9 (b) in Dembo and Zeitouni (1993). If $n > 1$, use (4.4) and induction. \square

Define probability measures \tilde{Q}_μ^α on H'_n by $\tilde{Q}_\mu^\alpha(dv) = (Z_\mu^\alpha)^{-1} \exp(v(\vec{f}_\mu)/\alpha) Q_\mu^\alpha(dv)$, where \vec{f}_μ was constructed in Lemma 5.1 and Z_μ^α is the normalizing constant. Let \tilde{J}_μ denote the functional $\tilde{J}_\mu(v) = J_\mu(v) - v(\vec{f}_\mu) + \log Z_\mu(\vec{f}_\mu)$, for $v \in H'_n$.

Lemma 5.2: *For $M, c > 0$ and $u \in H'_n$ let Γ denote the set of all $\mu \in M^+(\mathbb{R}^d)$ for which*

$\langle 1, \mu \rangle \leq M$ and $J_\mu(u) \leq c$. If $A \subset H'_n$ is closed, then, for all $N \geq 0$,

$$\overline{\lim}_{\alpha \downarrow 0} \sup_{\mu \in \Gamma} \left(\alpha \log \tilde{Q}_\mu^\alpha(A) + \inf_{v \in A} \tilde{J}_\mu(v) \wedge N \right) \leq 0.$$

Proof: By the last part of Lemma 5.1, the set $\{\vec{f}_\mu \mid \mu \in \Gamma\}$ is compact in H_n . Therefore, if $B \subset H_n$ is compact, $\{\vec{f}_\mu + \vec{f} \mid \mu \in \Gamma, f \in B\}$ is compact again, and Lemma 3.1 gives us that, as $\alpha \downarrow 0$,

$$(5.2) \quad \sup_{\vec{f} \in B, \mu \in \Gamma} \left| \alpha \log \int \exp\left(\frac{1}{\alpha} v(\vec{f})\right) \tilde{Q}_\mu^\alpha(dw) - \log \tilde{Z}_\mu(\vec{f}) \right| \longrightarrow 0,$$

where $\log \tilde{Z}_\mu(\vec{f}) = \log Z_\mu(\vec{f}_\mu + \vec{f}) - \log Z_\mu(\vec{f}_\mu)$, for $\vec{f} \in H_n$. For $v \in H'_n$ and $N \geq 0$, we can find $\vec{g}_{\mu,v} \in H_n$ such that $v(\vec{g}_{\mu,v}) - \log \tilde{Z}_\mu(\vec{g}_{\mu,v}) \geq \tilde{J}_\mu(v) \wedge N$. It can be seen easily that the mapping $\Gamma \ni \mu \mapsto \vec{g}_{\mu,v}$ can even be chosen to be continuous with compact image B in H_n . Hence, for $\delta > 0$, there is an open neighborhood $O_v \ni v$ such that

$$\sup_{\mu \in \Gamma, w \in O_v} |v(\vec{g}_{\mu,v}) - w(\vec{g}_{\mu,v})| \leq \delta.$$

Now, as usual, by the exponential Tschebyscheff inequality

$$\sup_{\mu \in \Gamma} \left(\alpha \log \tilde{Q}_\mu^\alpha(O_v) - \tilde{J}_\mu(v) \wedge N \right) \leq \delta + \sup_{\mu \in \Gamma} \left| \alpha \log \int e^{w(\vec{g}_{\mu,v})} \tilde{Q}_\mu^\alpha(dw) - \log \tilde{Z}_\mu(\vec{g}_{\mu,v}) \right|,$$

where the second term on the right hand side converges to zero by (5.2). The assertion for compact A now follows as in Dembo and Zeitouni (1993), p. 132, when using in addition the continuity of $\Gamma \ni \mu \mapsto \tilde{J}_\mu(v)$. Since $\Gamma \ni \mu \mapsto \log \tilde{Z}_\mu(\vec{f}) = \log Z_\mu(\vec{f}_\mu + \vec{f}) - \log Z_\mu(\vec{f}_\mu)$ is continuous by Lemma 5.1, it follows as in Dembo and Zeitouni (1993), p. 49 that, for each $L > 0$, there is a compact $K_L \subset H_n$ such that $\alpha \log \tilde{Q}_\mu^\alpha(K_L^C) \leq e^{-\alpha L}$, uniformly in $\mu \in \Gamma$ and for each $\alpha \in (0, 1]$. Standard arguments now show that the assertion holds for arbitrary closed sets A . \square

The proof of the next lemma is an easy modification of part b) of the proof of the Gärtner-Ellis theorem in Dembo and Zeitouni (1993), p. 49 ff.

Lemma 5.3: For given $M, c > 0$ and $u \in H'_n$, define Γ as in Lemma 5.2. Then, for every neighborhood U of u in H'_n , $\underline{\lim}_{\alpha \downarrow 0} \inf_{\mu \in \Gamma} (\alpha \log Q_\mu^\alpha(U) + J_\mu(u)) \geq 0$.

Proposition 5.4: *Suppose $\omega_0 \in C([0, 1]; M(\mathbb{R}^d))$ is such that $I_\mu(\omega_0) < c$, for some $\mu \in M^+(\mathbb{R}^d)$ and $c > 0$. Then, for any open neighborhood U of ω_0 , there is an open neighborhood $V \subset M^+(\mathbb{R}^d)$ such that $\mu \in V$ and*

$$\liminf_{\alpha \downarrow 0} \inf_{\nu \in V} \alpha \log \mathbb{P}_\nu[\widehat{X}^{\alpha, \beta(\alpha)} \in U] \geq -c.$$

Proof: Recall our convention that $\Omega = C([0, 1]; M(\mathbb{R}^d))$. By Lemma 4.1, we can assume that U is of the form

$$(5.3) \quad \bigcap_{i=1}^k \left\{ \omega \in \Omega \mid |\langle g_i, \omega(t) \rangle - \langle g_i, \omega_0(t) \rangle| < \varepsilon \forall t \right\},$$

for some $k \in \mathbb{N}$, $g_1, \dots, g_k \in BL(\mathbb{R}^d)$ and $\varepsilon > 0$. Now choose $\gamma < 1/2$, $L > 1$ and $M > \langle 1, \mu \rangle$. According to Lemma 2.1 there are $\alpha_0 > 0$ and $R < \infty$ such that, for all $\alpha \leq \alpha_0$ and ν with $\langle 1, \nu \rangle \leq M$, $\mathbb{P}_\nu[\widehat{X}^{\alpha, \beta(\alpha)} \notin K] \leq e^{-L/\alpha}$, where K is given by $K = \bigcap_{i=1}^k \{ \omega \in \Omega \mid |\langle g_i, \omega(\cdot) \rangle|_\gamma \leq R \}$, for $|\cdot|_\gamma$ denoting the Hölder norm of exponent γ as in (2.4). Clearly there is a finite partition $0 < t_1 < \dots < t_n \leq 1$ such that $\widetilde{U} \cap K \subset U$, where $\widetilde{U} := \bigcap_{i=1}^k \bigcap_{j=1}^n \{ \omega \mid |\langle g_i, \omega(t_j) \rangle - \langle g_i, \omega_0(t_j) \rangle| < \varepsilon/2 \}$. Then, for ν and α as above, $\mathbb{P}_\nu[\widehat{X}^{\alpha, \beta(\alpha)} \in U] \geq \mathbb{P}_\nu[\widehat{X}^{\alpha, \beta(\alpha)} \in \widetilde{U}] - e^{-L/\alpha}$. Now Lemma 5.3 applies to the term $\mathbb{P}_\nu[\widehat{X}^{\alpha, \beta(\alpha)} \in \widetilde{U}]$: Let H denote the linear hull of g_1, \dots, g_k , choose $c' \in (I_\mu(\omega_0), c)$ and let

$$\Gamma := \left\{ \nu \in M^+(\mathbb{R}^d) \mid \langle 1, \nu \rangle \leq M, I_\mu^{H, t_1 \dots t_n}(p_{H'_n}(\omega_0(t_1), \dots, \omega_0(t_n))) \leq c' \right\},$$

with $p_{H'_n}$ as in (4.1). Then it follows from (4.2), (4.3), (4.4) and the last part of Lemma 5.1 that μ is an interior point of Γ . In addition, Lemma 5.3 yields that

$$\liminf_{\alpha \downarrow 0} \inf_{\nu \in \Gamma} \alpha \log \mathbb{P}_\nu[\widehat{X}^{\alpha, \beta(\alpha)} \in \widetilde{U}] + c \geq 0.$$

This proves the assertion. \square

6. Proof of Theorem 1.2

For open sets $W, V_1, \dots, V_n \subset \Omega := C([0, 1]; M(\mathbb{R}^d))$, define a subset \mathcal{U} of \mathcal{A} by

$$\mathcal{U} = \left\{ B \in \mathcal{A} \mid B \subset W, B \cap V_1 \neq \emptyset, \dots, B \cap V_n \neq \emptyset \right\}.$$

The sets \mathcal{U} as above that contain K_μ form a base for the neighborhood system of K_μ in \mathcal{A} . See Castaing and Valadier (1977), Remark 2 in Section II.2 and the proof of Theorem

II-6, and note that there only the compactness of $K_0 := K_\mu$ is needed. Hence the proof of Theorem 1.2 reduces to showing that, for $W, V \subset \Omega$ open, the following two statements hold true

$$(6.1) \quad K_\mu \subset W \implies \Delta_\delta \subset W, \quad \text{for small } \delta, \mathbb{P}_\mu\text{-almost surely,}$$

$$(6.2) \quad K_\mu \cap V \neq \emptyset \implies \Delta_\delta \cap V \neq \emptyset, \quad \text{for all } \delta > 0, \mathbb{P}_\mu\text{-almost surely.}$$

Define two functions α and β by

$$(6.3) \quad \alpha(\varepsilon) = \frac{1}{2 \log \log \frac{1}{\varepsilon}} \quad (0 < \varepsilon < 1/e) \quad \text{and} \quad \beta(\alpha) = \frac{1}{\sqrt{\alpha \exp \exp \frac{1}{2\alpha}}} \quad (\alpha > 0).$$

Then, for ε as above,

$$(6.4) \quad \widehat{X}^{(\varepsilon)} := \widehat{X}^{\alpha(\varepsilon), \beta(\alpha(\varepsilon))} = \frac{\widehat{X}_\varepsilon}{\sqrt{2\varepsilon \log \log \varepsilon^{-1}}}.$$

Let us first prove (6.1). Observe that, by compactness of K_μ and by Lemma 4.1, there are $k \in \mathbb{N}$, $g_1, \dots, g_k \in BL(\mathbb{R}^d)$, and an open set $\widetilde{U} \subset C([0, 1]; \mathbb{R}^d)$ such that the open set $U \subset \Omega$ defined by

$$U = \left\{ \omega \in \Omega \mid (\langle g_1, \omega(\cdot) \rangle, \dots, \langle g_k, \omega(\cdot) \rangle) \in \widetilde{U} \right\}$$

satisfies $K_\mu \subset U \subset W$. By lower semicontinuity of I_μ , there is some λ such that $1/2 < \lambda < \inf_{\omega \notin U} I_\mu(\omega)$. Hence, by the upper bound of Theorem 1.1, $\mathbb{P}_\mu[\widehat{X}^{(\varepsilon)} \notin U] \leq \exp(-2\lambda \log \log \varepsilon^{-1})$, for all ε that are small enough. The Borel-Cantelli lemma hence gives us that, for any $\rho > 1$, $\{\widehat{X}^{(\rho^{-n})}, \widehat{X}^{(\rho^{-n-1})}, \dots\} \subset U$, for large n , \mathbb{P}_μ -a.s. But from this, one can already deduce that $\{\widehat{X}^{(\varepsilon)} \mid \varepsilon \leq \delta\} \subset U$, for small δ , \mathbb{P}_μ -a.s. See Deuschel and Stroock (1989), Lemma 1.4.3 and its proof. The claim (6.1) now follows from regularity of the topological space Ω .

To prove (6.2), observe first that $K_\mu \cap V \neq \emptyset$ for $V \subset \Omega$ open, implies the existence of some $\omega_0 \in V$ with $I_\mu(\omega_0) < 1/2$. Indeed, pick any $\omega_1 \in K_\mu \cap V$. Since $\lambda\omega_1 \rightarrow \omega_1$ as $\lambda \uparrow 1$, there is some $\lambda_0 < 1$ such that $\omega_0 := \lambda_0\omega_1 \in V$. But $I_\mu(\omega_0) = \lambda^2 I_\mu(\omega_1) < \lambda^2$. Now fix such an ω_0 . Then, by Lemma 4.1, we can find $k \in \mathbb{N}$, $\delta > 0$ and $g_1, \dots, g_k \in BL(\mathbb{R}^d)$ such that $U_{0,1}^\delta \subset V$, where

$$U_{s,t}^\delta := \bigcap_{i=1}^k \left\{ \omega \in \Omega \mid |\langle g_i, \omega(r) \rangle - \langle g_i, \omega_0(r) \rangle| < \delta, \quad s \leq r \leq t \right\},$$

for $0 \leq s \leq t \leq 1$. First we claim that there is an $\varepsilon > 0$ such that

$$(6.5) \quad \text{for } \mathbb{P}_\mu\text{-a.e. } \omega, \exists n_1 = n_1(\omega) \text{ such that } \widehat{X}^{(\varepsilon^n)}(\omega) \in U_{0,\varepsilon}^\delta \text{ for all } n \geq n_1.$$

Indeed, define

$$\Phi_t(\omega) = \max_{i=1,\dots,k} \sup_{0 \leq r \leq t} \left| \langle g_i, \omega(r) \rangle - \langle g_i, \omega_0(r) \rangle \right|,$$

for $t \geq 0$ and $\omega \in \Omega$. Then each Φ_t is continuous and decreases to zero as $t \downarrow 0$. By Dini's theorem this convergence is even uniform on K_μ . Hence there is some $\varepsilon > 0$ such that $K_\mu \subset U_{0,\varepsilon}^\delta$. Thus (6.5) follows from (6.1).

Now fix ε as above and introduce σ -algebras $\mathcal{F}_n := \sigma(X_s \mid 0 \leq s \leq \varepsilon^n)$, so that $\widehat{X}^{(\varepsilon^n)}$ is \mathcal{F}_n -measurable. Using the Markov property of X , we get that, for \mathbb{P}_μ -almost every $\omega \in \Omega$,

$$\begin{aligned} \mathbb{P}_\mu \left[\widehat{X}^{(\varepsilon^n)} \in U_{0,1}^\delta \mid \mathcal{F}_{n+1} \right] (\omega) &= \mathbb{P}_\mu \left[\widehat{X}^{(\varepsilon^n)} \in U_{0,\varepsilon}^\delta \cap U_{\varepsilon,1}^\delta \mid \mathcal{F}_{n+1} \right] (\omega) \\ &= \mathbf{1}_{U_{0,\varepsilon}^\delta}(\widehat{X}^{(\varepsilon^n)}(\omega)) \cdot \mathbb{P}_{X_{\varepsilon^{n+1}}(\omega)} \left[\widehat{X}^{(\varepsilon^n)} + (X_{\varepsilon^{n+1}}(\omega) - \mu P_{\varepsilon^{n+1}})P. \in U_{0,1-\varepsilon}^\delta \right]. \end{aligned}$$

Recall that $\mathbf{1}_{U_{0,\varepsilon}^\delta}(\widehat{X}^{(\varepsilon^n)}(\omega)) = 1$ for all $n \geq n_1$ by (6.5).

As a next step note that, for \mathbb{P}_μ -almost every ω , there is some $n_2 = n_2(\omega)$ such that

$$\mathbb{P}_{X_{\varepsilon^{n+1}}(\omega)} \left[\widehat{X}^{(\varepsilon^n)} + (X_{\varepsilon^{n+1}}(\omega) - \mu P_{\varepsilon^{n+1}})P. \in U_{0,1-\varepsilon}^\delta \right] \geq \mathbb{P}_{X_{\varepsilon^{n+1}}(\omega)} \left[\widehat{X}^{(\varepsilon^n)} \in U_{0,1}^{\delta/2} \right]$$

for all $n \geq n_2$. This follows from the Feller property of $(P_t)_{t \geq 0}$ and from $X_t \rightarrow \mu$ as $t \downarrow 0$. Now recall that $I_\mu(\omega_0) < 1/2$. We hence can apply Proposition 5.4 to get the existence of some $n_3 = n_3(\omega)$ such that

$$\mathbb{P}_{X_{\varepsilon^{n+1}}(\omega)} \left[\widehat{X}^{(\varepsilon^n)} \in U_{0,1}^{\delta/2} \right] \geq e^{-\alpha(\varepsilon^n)/2} = \frac{1}{n \log \varepsilon^{-1}} \quad \forall n \geq n_3.$$

Putting all these estimates together, we finally arrive at

$$\mathbb{P}_\mu \left[X^{(\varepsilon^n)} \in U_{0,1}^\delta \mid \mathcal{F}_{n+1} \right] (\omega) \geq \frac{1}{n \log \varepsilon^{-1}} \quad \forall n \geq n_1 \vee n_2 \vee n_3,$$

for \mathbb{P}_μ -almost every ω , and hence

$$\sum_{n=1}^{\infty} \mathbb{P}_\mu \left[X^{(\varepsilon^n)} \in U_{0,1}^\delta \mid \mathcal{F}_{n+1} \right] (\omega) = \infty.$$

Therefore (6.2) follows from the next lemma, that can be proved as Theorem 46 in Chapter V of Dellacherie and Meyer (1980), when using martingale convergence for *decreasing* σ -fields.

Lemma 6.1: *Suppose (Ω, \mathcal{F}, P) is a probability space and $(\mathcal{F}_n)_{n \in \mathbb{N}}$ are σ -fields such that $\mathcal{F} \supset \mathcal{F}_0 \supset \mathcal{F}_1 \supset \dots$. If $(A_n)_{n \in \mathbb{N}}$ is a sequence of events with $A_n \in \mathcal{F}_n$, for each n , then, modulo P -nullsets,*

$$\left\{ \sum_{n=0}^{\infty} P[A_n | \mathcal{F}_{n+1}] = \infty \right\} = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} A_n.$$

7. An embedding result for Hölder-Orlicz spaces

This section is taken from the author's thesis Schied (1995). For some $\gamma > 0$ and any map $w \in C[0, 1]^n := C([0, 1]^n; \mathbb{R})$ let

$$(7.1) \quad |w|_{\gamma} = \sup \left\{ \frac{|w(t) - w(s)|}{|t - s|^{\gamma}} \mid s, t \in [0, 1]^n, s \neq t \right\}$$

denote the Hölder norm of w with exponent γ . The Hölder space $H^{\gamma}[0, 1]^n$ consists of all maps w with finite Hölder norm. More generally, for any normed space E , we can define the space $H^{\gamma}([0, 1]^n; E)$ of E -valued Hölder continuous mappings. In his well known criterion, Kolmogorov uses a moment condition to establish Hölder continuity of the sample paths of a stochastic process. Here we are going to give conditions, which guarantee that the Hölder norm of the sample paths even possesses exponential moments. Our main result is the following corollary to Proposition 7.2 below.

Corollary 7.1: *Suppose ξ is a continuous random field with parameter range $[0, 1]^n$ for which there exist positive constants γ , λ and κ such that*

$$E \left[\exp \left(\frac{\lambda}{|t - s|^{\gamma}} |\xi_t - \xi_s| \right) \right] \leq \kappa \quad (s, t \in [0, 1]^n, s \neq t).$$

Then, for any $\gamma' < \gamma$,

$$P \left[|\xi|_{\gamma'} \geq L \right] \leq (1 + \kappa) \exp \left(\frac{-L\lambda}{c_n} \right),$$

where the constant c_n may be chosen in such a way that it only depends on γ , γ' and n :

$$(7.2) \quad c_n = \frac{n(1 + p_0!) \cdot 2^{\gamma'+1}}{1 - 2^{-p_0^{-1}((\gamma - \gamma')p_0 - n)}} \quad \text{with } p_0 := \left[\frac{n}{\gamma - \gamma'} \right] + 1$$

and with $[\cdot]$ denoting integer part.

For $\kappa > 0$, let Φ_κ denote the function $\Phi_\kappa(x) = (e^x - 1)/\kappa$. The Luxemburg norm of a measurable function f on Ω with respect to Φ_κ then is given as usual by

$$\|f\|_{\Phi_\kappa} = \inf \left\{ \beta > 0 \mid \int \Phi_\kappa \left(\frac{|f|}{\beta} \right) dP \leq 1 \right\}.$$

Note that, if we replaced Φ_κ by the mapping $x \mapsto x^p$, we would get the classical L^p -norm $\|f\|_p$. The space $L_{\Phi_\kappa}(\Omega, P)$ of all measurable functions on Ω with finite Luxemburg norm is usually called the Orlicz space with respect to Φ_κ . Here too, we can analogously define Orlicz spaces $L_{\Phi_\kappa}(\Omega; E, P)$ of mappings taking their values in some normed space E .

Proposition 7.2: *Let $\kappa \geq 1$, $0 < \gamma' < \gamma$ and let $\|\cdot\|_{\kappa, \gamma}$ denote the norm in the space $H^\gamma([0, 1]^n; L_{\Phi_\kappa}(\Omega, P))$. Then every process $\xi \in H^\gamma([0, 1]^n; L_{\Phi_\kappa}(\Omega, P))$ possesses a unique modification $\tilde{\xi}$ lying in $L_{\Phi_\kappa}(\Omega; H^{\gamma'}[0, 1]^n, P)$. Moreover $\tilde{\xi}$ satisfies*

$$\|\tilde{\xi} \cdot |_{\gamma'}\|_{\Phi_\kappa} \leq c_n \cdot \|\xi\|_{\kappa, \gamma},$$

where c_n can be chosen as in (7.2). I.e., we have the continuous embedding

$$H^\gamma([0, 1]^n; L_{\Phi_\kappa}(\Omega, P)) \hookrightarrow L_{\Phi_\kappa}(\Omega; H^{\gamma'}[0, 1]^n, P).$$

Remark: The classical Kolmogorov criterion can be stated in the same fashion as Proposition 7.2. In fact, inspecting the proof of Theorem 2.1 in Revuz and Yor (1994), p. 25, one finds that

$$H^{\gamma+n/p}([0, 1]^n; L^p(\Omega, P)) \hookrightarrow L^p(\Omega; H^{\gamma'}[0, 1]^n, P),$$

for all $p \geq 1$ and $0 < \gamma' < \gamma$.

Proof of Proposition 7.2: Let $H := \|\xi\|_{\kappa, \gamma}$. Then $\|\xi_t - \xi_s\|_{\Phi_\kappa} \leq H \cdot |t - s|^\gamma$. Since $\Phi_\kappa(x) \geq x^p/(\kappa p!)$ ($x \geq 0$, $p \in \mathbb{N}$), we get that $\|\cdot\|_{\Phi_\kappa} \geq (\kappa p!)^{-1/p} \|\cdot\|_p$ with $\|\cdot\|_p$ denoting the usual L^p -norm. Hence $E[|\xi_t - \xi_s|^p] \leq p! \kappa H^p |t - s|^{p\gamma}$. Now we follow Deuschel and Wang (1993). For $p \geq p_0$, a version of the classical Kolmogorov criterion implies the existence of a continuous modification $\tilde{\xi}$, satisfying $E[|\tilde{\xi} \cdot |_{\gamma'}^p] \leq p! \kappa (\tilde{c}_n H)^p$ with the constant

$$\tilde{c}_n = \frac{2^{\gamma'+1}}{1 - 2^{-p_0^{-1}((\gamma-\gamma')p_0-n)}}.$$

Cf. Theorem 2.1 in Revuz and Yor (1994), p. 25 and its proof. In the case $p < p_0$ Jensen's inequality yields that

$$E[|\tilde{\xi} \cdot |_{\gamma'}^p] \leq E[|\tilde{\xi} \cdot |_{\gamma'}^{p_0}]^{\frac{p}{p_0}} \leq (p_0! \kappa)^{\frac{p}{p_0}} (\tilde{c}_n H)^p \leq p_0! \kappa \tilde{c}_n^p H^p.$$

By summing up, we can now extend these estimates to our Orlicz spaces. Indeed we get that, for any $\tilde{c}_n H > \beta > 0$,

$$E \left[\Phi_\kappa \left(\frac{|\tilde{\xi} \cdot |_{\gamma'}}{\beta} \right) \right] \leq p_0! \sum_{k=1}^{p_0-1} \frac{1}{k!} \left(\frac{\tilde{c}_n H}{\beta} \right)^k + \sum_{k=p_0}^{\infty} \left(\frac{\tilde{c}_n H}{\beta} \right)^k \leq p_0! \frac{\tilde{c}_n H}{\beta - \tilde{c}_n H}.$$

Now the right hand side is less than one, if $\beta \geq (p_0! + 1) \cdot \tilde{c}_n \cdot H$. So by definition of the Luxemburg norm we conclude that $\| |\tilde{\xi} \cdot |_{\gamma'} \|_{\Phi_\kappa} \leq (p_0! + 1) \cdot \tilde{c}_n \cdot H$, and Proposition 7.2 is proved. \square

Proof of Corollary 7.1: Our assumption implies that $E \left[\Phi_\kappa (\lambda |t-s|^{-\gamma} |\xi_t - \xi_s|) \right] \leq 1$, and hence that $\| |\xi \cdot |_{\kappa, \gamma} \| \leq \lambda^{-1}$. Now we can apply Proposition 7.2 to get that $\| |\xi \cdot |_{\gamma'} \|_{\Phi_\kappa} \leq c_n / \lambda$, for any $\gamma' < \gamma$. Consequently

$$P \left[|\xi \cdot |_{\gamma'} > L \right] \leq E \left[\exp \left(\frac{\lambda |\xi \cdot |_{\gamma'}}{c_n} \right) \right] \exp \left(-\frac{L\lambda}{c_n} \right) \leq (\kappa + 1) \exp \left(\frac{-L\lambda}{c_n} \right).$$

Hence the corollary is proved. \square

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