Robustness of Delta hedging for path-dependent options in local volatility models

Alexander Schied*
TU Berlin, Institut für Mathematik, MA 7-4
Strasse des 17. Juni 136
10623 Berlin, Germany
e-mail: schied@math.tu-berlin.de

Mitja Stadje
Princeton University, ORFE Dept.
Engineering Quadrangle
Princeton, NJ 08544
e-mail: mstadje@princeton.edu

Abstract: We consider the performance of the Delta hedging strategy obtained from a local volatility model if we use as input the physical prices instead of the model price process. This hedging strategy is called robust if it yields a superhedge as soon as the local volatility model overestimates the market volatility. We show that robustness holds for a standard Black & Scholes model whenever we hedge a path-dependent derivative with a convex payoff function. In a genuine local volatility model, the situation is shown to be less stable: robustness can break down for many relevant convex payoffs including average strike Asian options, lookback puts, floating strike forward starts, and their aggregated cliquets. We prove furthermore that a sufficient condition for the robustness in every local volatility model is the directional convexity of the payoff function.

1 Introduction

One of the key features of local volatility models is their completeness: in the model world, every contingent claim admits a perfect hedge in terms of its Delta hedging strategy. But what is the performance of this Delta hedge if we use as input the *physical* prices quoted at the stock exchange instead of the model price process? Following El Karoui et al. [6], we will say that the Delta hedging strategy is *robust* if this physical Delta hedge is a superhedge for the claim as soon as the local volatility model overestimates the market volatility.

In the case of a European option $h(S_T)$ it was shown in [6] that the convexity of h is a sufficient condition for the robustness of the Delta hedge in any reasonable local volatility model. This result is closely related to volatility comparison techniques as introduced by Hajek [13] and El Karoui et al. [6]. These techniques have since been generalized to multivariate price processes and processes with jumps; see Gushchin and Mordecki [11], Bergenthum and Rüschendorf [1], Hobson [15], Janson and Tysk [16, 17], Ekström et al.

*Supported by Deutsche Forschungsgemeinschaft through the Research Center MATHEON "Mathematics for key technologies" (FZT 86).

[5] and the references therein. See also Lyons [19] for an analytic, "probability-free" result. It relies on a setup for the physical price process, which is based on Föllmer's pathwise Itô formula [7] and in this sense is similar to ours. Volatility comparison techniques are also known to work in the context of American options [6] and can be translated to drift comparison via the Girsanov transform [20]. While many authors have used this technique to obtain ordering results for *prices* of contingent claims, our emphasis will rather be on *hedging* issues.

In this paper, we study the robustness of the Delta hedging strategy for general *path-dependent* derivatives of the form $h(S_{t_1},\dots,S_{t_n})$. We show in Corollary 2.4 that the convexity of h is a sufficient condition for robustness if the local volatility does not depend on the stock, i.e., if we are working with a Black & Scholes model. This positive result applies to a large number of standard path-dependent options including average-strike and average-price Asian options, lookback puts, floating strike forward starts, and their aggregated cliquets. Perhaps this fact might help to explain the good hedging performance of the Black & Scholes model, which is sometimes reported by practitioners.

In a genuine local volatility model, however, the situation can be more complicated and less stable. In Theorem 2.5, we consider a reasonable family of local volatility models, in which robustness breaks down for many relevant path-dependent derivatives including average-strike Asian options, lookback puts, floating strike forward starts, and their aggregated cliquets, even though they all correspond to convex payoff functions. The volatility functions for which our result is valid include a frequently observed pattern of empirical local volatilities in many equity markets.

In Theorem 2.6, we give a sufficient criterion on the payoff function h, under which the Delta hedging strategy is robust for any reasonable local volatility model. This condition is the directional convexity of h, which is also analyzed by Bergenthum and Rüschendorf [1] in a multivariate though single-time setting. Directional convexity neither implies nor is implied by convexity in the usual sense. It applies, for instance, to the payoff function of an average price Asian call.

In Section 2, we describe our setup and state our main results. Most proofs are deferred to Section 3.

2 Statement of results

A common approach to valuing exotic derivatives is to use a diffusion process $S=(S_t)_{t\geq 0}$ based on local volatility for modeling the risk-neutral evolution of the forward price of the underlying. The term 'local volatility' means that the instantaneous volatility at time t is given as a function $\sigma(t,S_t)$ of t and S_t alone. That is, S is a solution of the stochastic differential equation

$$dS_t=\sigma(t,S_t)S_t\,dW_t$$

for a standard Brownian motion W. For simplicity, we will assume henceforth that the risk-free interest rate and all dividend payments are zero, so that we need not distinguish between the underlying stock and its forward price. This is possible without loss of generality if the discount factor is deterministic.

In practice, the volatility function $\sigma(t,x)$ is often chosen in such a way that the local volatility model is calibrated to the market prices of liquid plain vanilla options. Calibration can be achieved by combining the Dupire formula [4] (see also Gyöngy [12] for related results) with an appropriate interpolation method. This approach guarantees that all European derivatives with payoff $h(S_t)$ are priced consistently with all plain vanilla instruments.

Once the local volatility model is set up, it is typically used for the analysis of path-dependent exotic derivatives. In practice, the payoff of such a derivative is of the form

$$H(S) = h(S_{t_1}, \dots, S_{t_n})$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n \le T$ and $h \ge 0$. In this paper we will mainly—but not exclusively—be interested in derivatives whose payoff function h is *convex*. Standard examples include average price Asian put and call options,

$$\Big(K - \frac{1}{n}\sum_{i=1}^{n} S_{t_i}\Big)^{+}, \qquad \Big(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\Big)^{+}, \tag{1}$$

average strike Asians,

$$\Big(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - S_T\Big)^{+}, \qquad \Big(S_T - \frac{1}{n}\sum_{i=1}^{n} S_{t_i}\Big)^{+}, \tag{2}$$

lookback put options,

$$\max_{i=1,\dots,n} S_{t_i}\,, \tag{3}$$

floating-strike forward starting put and call options,

$$(KS_{T_0} - S_T)^{+}, \qquad (S_T - KS_{T_0})^{+} \qquad \text{for } 0 < T_0 < T, \tag{4}$$

and their aggregated cliquets,

$$\sum_{i=1}^{n}(KS_{t_{i-1}} - S_{t_i})^{+}. \qquad \sum_{i=1}^{n}(S_{t_i} - KS_{t_{i-1}})^{+}. \tag{5}$$

In the sequel, we will distinguish between the *model* S_t of the stock price and the actual *data* quoted at the stock exchange. This *physical price* quoted at time t will be denoted by X_t. We assume that such a price X_t is available at any time $t \in [0,T]$. Since there is only a single quote of the stock at a given time and no control experiment is possible, it is natural to think of $X = (X_t)_{0\le t\le T}$ as a *fixed function* on the time interval $[0,T]$ and not as a stochastic process involving additional randomization. In doing so, we follow the ideas of Föllmer [8, 9]. Thus, for a derivative with payoff function h, we denote its physical payoff by $h(X_{t_1}, \dots, X_{t_n})$, while it is modeled as the random variable

$h(S_{t_1},\dots,S_{t_n})$ in the local volatility model. It will sometimes be convenient to write $h(X_{t_1},\dots,X_{t_n}) = H(X)$ and $h(S_{t_1},\dots,S_{t_n}) = H(S)$.

So let $h : [0,\infty)^n \to [0,\infty)$ be the payoff function for the path-dependent claim H. At times $t \in [0,t_1)$, the value function of $H(S)$ will be of the form

$$v(t,x) = E[\,H(S)\,|\,S_t = x\,] = E[\,h(S_{t_1},\dots,S_{t_n})\,|\,S_t = x\,].$$

By taking $t = 0$ and $x = X_0$, this gives the *price* of the derivative in the local volatility model:

$$v(0,X_0) = E[\,H(S)\,|\,S_0 = X_0\,].$$

At a later stage, for $t \in [t_k,t_{k+1})$, the fixings $X_{t_1},\dots,X_{t_k}$ of the observed market prices will have been locked in as additional parameters of the value function:

$$\begin{aligned} v(t,X_{t_1},\dots,X_{t_k},x) &= E[\,H(S)\,|\,S_{t_1} = X_{t_1},\dots,S_{t_k} = X_{t_k}, S_t = x\,] \\ &= E[\,h(X_{t_1},\dots,X_{t_k},S_{t_{k+1}},\dots,S_{t_n})\,|\,S_t = x\,]. \end{aligned}$$

Here, the second identity follows from the Markov property of S. For $t \geq t_n$, we will finally have

$$v(t,X_{t_1},\dots,X_{t_n},x) = h(X_{t_1},\dots,X_{t_n}) = H(X) \qquad \text{for all } x.$$

In analogy to our notation $H(X)$, we will use the shorthand notation

$$v(t,X) := v(t,X_{t_1},\cdots,X_{t_k},X_t) \qquad \text{for } t_k \leq t < t_{k+1},$$

for the path-dependent value function. Similarly, we define $v(t,S)$ and we will use the same notation on other functions such as the derivatives of v.

Note that the local volatility model is a complete market model. Thus, the option's payoff in the local volatility model can be represented in terms of the corresponding Delta hedging strategy:

$$H(S) = v(0,S_0) + \int_0^T v_x(t,S)\,dS_t \qquad P\text{-a.s.} \tag{6}$$

Here and in the sequel,

$$v_x(t,S) = v_x(t,S_{t_1},\dots,S_{t_k},S_t) \qquad \text{for } t_k \leq t < t_{k+1}. \tag{7}$$

will be shorthand for the derivative of the value function $v(t,x_1,\dots,x_k,x)$ with respect to its final argument x, and we assume that this derivative is well-defined for a.e. t. Similarly, we will use the notations v_{xx} and v_t.

The question we are interested in here is whether hedging with the Delta obtained from the local volatility model also works for the physical price process X. That is, can we hedge the *physical* payoff $H(X)$ by replacing in (6) and (7) the model process (S_t) with the real-world prices $(X_t)_{0\leq t\leq T}$?

The corresponding self-financing hedging strategy will then involve the price $v(0,X_0)$, and the number of shares held at time t will be determined by $v_x(t,X)$. If trading occurs at finitely many times $s_1 < \cdots < s_k$ taken from a partition $\zeta_n = \{0, s_1, \dots, s_k\} \subset [0,T]$, then the corresponding value process will be

$$v(0,X_0) + \sum_{s_i \in \zeta_n} v_x(s_{i-1}, X)(X_{s_i} - X_{s_{i-1}}). \tag{8}$$

If we can pass to a limit in (8) for a fixed sequence $\zeta_1 \subset \zeta_2 \subset \cdots$ of partitions whose meshes tend to zero then the value process takes the form

$$v(0,X_0) + \int_0^T v_x(t,X)\,dX_t,$$

where the integral on the right is the *pathwise Itô integral* defined in the sense of Föllmer [7]:

$$\int_0^T v_x(t,X)\,dX_t = \lim_{n\uparrow\infty} \sum_{s_i \in \zeta_n} v_x(s_{i-1}, X)(X_{s_i} - X_{s_{i-1}}).$$

According to Föllmer [7], this passage to the limit is possible if the path $t \mapsto X_t$ is continuous, has a continuous *pathwise quadratic variation*,

$$\langle X \rangle_t := \lim_{n\uparrow\infty} \sum_{\substack{s_i \in \zeta_n \\ s_i \le t}} (X_{s_i} - X_{s_{i-1}})^2,$$

and the value function v is sufficiently regular. Thus, we will henceforth assume that X satisfies the above conditions, and we point out once more that no stochastic model for X is needed. The main regularity conditions for v can be deduced from the following regularity assumptions on the local volatility function and the payoff function h; see Proposition 2.2 below. These assumptions are almost identical to the ones in [6]. Some of them can be relaxed at the expense of tightening others; see the Remark at the end of this section. Since local volatility functions in practise typically arise as the interpolation of discrete values obtained from a discretized version of Dupire's formula, there is no loss of generality from a practical point of view in assuming smoothness and boundedness of our local volatility function.

Assumption 2.1 *Throughout this paper, we will assume that the local volatility function $\sigma(t,x)$ satisfies the following conditions:*

(a) $\sigma : [0,T] \times (0,\infty) \to (0,\infty)$ *is continuously differentiable, bounded, and bounded away from zero.*

(b) $\sigma(t,x)x$ *is Lipschitz continuous in x, uniformly in $t \in [0,T]$.*

We will also assume henceforth that h denotes a continuous function from $[0,\infty)^n$ to $[0,\infty)$ that satisfies the polynomial growth condition

$$h(x) \le C(1+|x|^p), \qquad x \in [0,\infty)^n,$$

for certain constants $C, p \ge 0$. Such a function will be called a payoff function.

Remark. Assumption 2.1 has some immediate consequences. First, the boundedness of σ implies via Novikov's theorem that S is a strictly positive martingale with finite moments $E[\, S_t^p \mid S_0 = x\,]$ for all $p \in [0,\infty)$. In particular, the value function associated with any payoff function h is finite. Second, condition (b) ensures that the stochastic differential equation $dS_t = \sigma(t, S_t) S_t \, dW_t$ admits a strong solution, which is pathwise unique, a property which is also important in practice to secure the convergence of numerical algorithms. ♢

As long as the intermediate arguments $X_{t_1}, \dots, X_{t_k}$ do not matter, we may only spell out the dependence of v (and its derivatives) on its first and last arguments t and x by using the shorthand notation

$$v(t, \cdot, x) := v(t, \underbrace{\cdot, \cdots, \cdot}_{k \text{ arguments}}, x) \qquad \text{for } t_k \le t < t_{k+1}.$$

We turn to a first result, which states the technical regularity properties of our value function needed for the definition of the Delta hedging strategy.

Proposition 2.2 (Regularity of the value function) *Suppose that h is a payoff function. Then $v(t, \cdot, x)$ is continuous on $[0,T] \times [0,\infty)$, continuously differentiable in $t \in \bigcup_k (t_k, t_{k+1})$, twice continuously differentiable in $x \in (0,\infty)$, and satisfies the partial differential equation*

$$v_t(t, \cdot, x) + \frac{1}{2}\sigma(t,x)^2 x^2 v_{xx}(t, \cdot, x) = 0, \qquad t \in \bigcup_k (t_k, t_{k+1}),\ x \in (0,\infty). \tag{9}$$

If moreover

$$\int_0^T |v_{xx}(t, X)| \, d\langle X\rangle_t + \int_0^T |v_t(t, X)| \, dt < \infty, \tag{10}$$

then the pathwise Itô integral $\int_0^T v_x(t, X)\, dX_t$ is well-defined, and Itô's formula holds:

$$v(T, X) = v(0, X) + \int_0^T v_x(t, X)\, dX_t + \frac{1}{2}\int_0^T v_{xx}(t, X)\, d\langle X\rangle_t + \int_0^T v_t(t, X)\, dt. \tag{11}$$

The condition (10) is clearly necessary as to make sense of the right-hand side of (11) and in turn of the Delta hedging strategy. It is satisfied as soon as h belongs to C^2.

Let us return to our problem of hedging the physical payoff $H(X)$ of a path-dependent option with the Delta hedging strategy obtained from the local volatility model. The trader carrying out the hedge will insert at each time t the market spot price X_t into the local volatility model. In particular, $\sigma(t, X_t)$ will serve as an estimate for the *short volatility* at time t. Thus, we will say that σ *overestimates the market volatility* if

$$\int_s^t \sigma^2(r, X_r) X_r^2 \, dr \ge \langle X\rangle_t - \langle X\rangle_s \qquad \text{for all } 0 \le s < t \le T. \tag{12}$$

If the reverse inequality holds, we will say that σ *underestimates the market volatility.*

Remark. If inequality (12) holds, then the function $t \mapsto \langle X\rangle_t$ is absolutely continuous and can hence be written as $\langle X\rangle_t = \int_0^t \varsigma_s X_s^2\,ds$ for some function $\varsigma \geq 0$, which can be interpreted as the *short variance* of the market. In this case, (12) is equivalent to requiring

$$\sigma(t, X_t) \geq \sqrt{\varsigma_t} \qquad \text{for a.e. } t \in [0,T] \text{ such that } X_t > 0.$$

♢

Definition (Robustness of the Delta hedging strategy) Let $H(X) = h(X_{t_1}, \dots, X_{t_n})$ be the payoff of a path-dependent derivative satisfying (10). We will say that the Delta hedging strategy for H obtained from the local volatility model is *robust* if the following two conditions hold: If σ overestimates the market volatility, then there is a superhedge for the seller in the sense that

$$v(0, X_0) + \int_0^T v_x(t, X)\,dX_t \geq H(X); \tag{13}$$

if σ underestimates the market volatility, then there is a superhedge for the buyer:

$$v(0, X_0) + \int_0^T v_x(t, X)\,dX_t \leq H(X).$$

The preceding notion of robustness is due to El Karoui et al. [6]. If a Delta hedging strategy is robust, then a trader can monitor its performance by comparing $\sigma(t, X_t)$ to the realized market volatility.

Remark (Volatility comparison for prices) While our notion of robustness is understood in a strictly pathwise sense, some other authors emphasize its impact on option *pricing* rather than hedging. To this end, one assumes that the market prices X are a particular realization of the sample paths of a continuous local martingale. Then, if σ overestimates the market volatility almost surely and if the Delta hedging strategy is a supermartingale, taking expectations in (13) yields

$$v(0, X_0) = E[\, H(S) \,|\, S_0 = X_0 \,] \geq E[\, H(X) \,].$$

This result can be interpreted as an ordering between the price computed in the local volatility model and the "true" market price of the derivative. This latter price, however, is not observable unless the derivative is liquidly traded. ♢

We can now state our first basic hedging result. It relates the robustness of the Delta hedging strategy to the positivity of the corresponding Gamma v_{xx}. Note that the positivity of the Gamma will follow automatically if the function $x \mapsto v(t, \cdot, x)$ is convex for all t.

Proposition 2.3 (Positivity of the Gamma implies robustness) *Suppose h is a payoff function such that the option's Gamma satisfies $v_{xx}(t, X) \geq 0$ for a.e. t and such that* (10) *holds. Then the Delta hedging strategy is robust.*

Proof: The proof is short and instructive. It relies on arguments from El Karoui et al. [6]. By Proposition 2.2 and since $v(T,X) = H(X)$,

$$v(0,X) + \int_0^T v_x(t,X)\,dX_t = H(X) - \frac{1}{2}\int_0^T v_{xx}(t,X)\,d\langle X\rangle_t - \int_0^T v_t(t,X)\,dt. \qquad (14)$$

Now suppose that σ overestimates the market volatility. If $v_{xx}(t,X) \geq 0$, then we get $\int_0^T v_{xx}(t,X)\,d\langle X\rangle_t \leq \int_0^T v_{xx}(t,X)\sigma(t,X_t)^2X_t^2\,dt$. The PDE (9) then implies that the right-hand side of (14) dominates $H(X)$, i.e., we have a successful seller's hedge. The argument for the buyer's hedge is identical. □

Our next result shows that the Delta hedging strategy obtained from a Black & Scholes model will be robust for *all* convex payoff functions. This class includes in particular all Asian options, lookback puts, floating strike forward starts, and their aggregated cliquets. Perhaps this fact might help to explain the good hedging performance of the Black & Scholes model, which is sometimes reported by practitioners.

Corollary 2.4 (Robustness in a Black & Scholes model) *Suppose that the local volatility function $\sigma(t,x)$ does not depend on x. Then the value function of any convex payoff function is again convex. In particular, the corresponding Delta hedging strategy is robust as soon as* (10) *holds.*

In a genuine local volatility model, it is known from El Karoui et al. [6] that all European-style derivatives $h(S_T)$ with a convex payoff function h have convex payoff functions and thus admit robust Delta hedges. For *path-dependent* derivatives, however, the situation is more complicated and less stable: The value functions of a large number of fairly standard path-dependent derivatives may not be convex and Delta hedging may not be robust, even though they correspond to convex payoff functions. This is illustrated by our next result. Its conditions apply in particular to average strike Asian options, lookback puts, floating strike forward starts, and their aggregated cliquets.

Theorem 2.5 (Non-robustness for standard derivatives) *Suppose that h is a convex payoff function that is not identically equal to a linear functional and is positively homogeneous:*

$$h(zx_1,\ldots,zx_n) = zh(x_1,\ldots,x_n) \qquad \textit{for } z \geq 0.$$

Suppose moreover that there are constants $0 < c < C$ and $0 < x_0 < x_1$ such that the local volatility function $\sigma(\cdot)$ satisfies $\sigma(t,x) \geq C$ if $0 \leq x \leq x_0$ and $\sigma(t,x) \leq c$ if $x_1 \leq x < \infty$. Then the corresponding value function is not convex and there exists a path X along which the Delta hedging strategy is not robust.

In the preceding theorem, the particular shape of $\sigma(t,x)$ for $x_0 < x < x_1$ can be arbitrary. The assumed property is in fact a frequently observed empirical pattern of local volatility functions in many equity markets.

Our final result gives a sufficient condition on a payoff function h, which guarantees that the corresponding value function is convex in its final argument for any local volatility model. Thus, the Delta hedging strategy of such a payoff will be robust. Our condition is the *directional convexity* of h, that is, all marginal functions $x_i \mapsto h(x_1, \dots, x_i, \dots, x_n)$ are convex and all right-hand derivatives

$$h^+_{x_i}(x_1, \dots, x_n) := \lim_{\delta \downarrow 0} \frac{1}{\delta}\big(h(x_1, \dots, x_i + \delta, \dots, x_n) - h(x_1, \dots, x_i, \dots, x_n)\big)$$

are increasing with respect to each component x_j. Within our example class (1)–(5), this condition is satisfied by the average price Asian call options,

$$\Big(\frac{1}{n}\sum_{i=0}^{n} S_{t_i} - K\Big)^+.$$

Note that directional convexity neither implies nor is implied by convexity in the usual sense: there are non-convex but directionally convex payoff functions such as the one corresponding to a 'geometric Asian call',

$$\Big(\prod_{i=0}^{n} S_{t_i} - K\Big)^+.$$

Theorem 2.6 (Directional convexity implies robustness) *Let h be a directionally convex payoff function. Then, for $t_{k-1} \le t < t_k$, the value function $v(t, x_1, \dots, x_k)$ is also directionally convex in $x_1, \dots, x_k$. In particular, $x \mapsto v(t, \cdot, x)$ is convex and the corresponding Delta hedging strategy is robust as soon as* (10) *holds.*

Remark (On the standard definition of directional convexity) The standard definition of directional convexity of a function $f : \mathbb{R}^n \to \mathbb{R}$ requires that

$$x_j \mapsto f(x_1, \dots, x_i + \varepsilon, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)$$

is an increasing function whenever $\varepsilon > 0$ and $i, j \in \{1, \dots, n\}$; see [1] and the references therein. Taking $j = i$ implies that the function $g(x_i) := f(x_1, \dots, x_i, \dots, x_n)$ is *Wright convex*, i.e., $x \mapsto g(x + \varepsilon) - g(x)$ is increasing for each $\varepsilon > 0$. As observed by Wright [22], this immediately implies that

$$g(x) + g(y) \ge 2g\Big(\frac{x+y}{2}\Big).$$

It is well known that the preceding inequality implies convexity if g is continuous; see, e.g., Theorem 86 in Hardy et al. [14]. Hence, we can conclude that our definition of directional convexity coincides with the usual one as soon as $x_i \mapsto f(x_1, \dots, x_i, \dots, x_n)$ is continuous for each i. However, on p. 96 of [14], Hardy, Littlewood, and Polya construct a discontinuous function g such that $g(x + y) = g(x) + g(y)$. Such a discontinuous but additive g is clearly Wright convex but not convex in the usual sense. Thus, our definition of directional convexity is generally stronger than the standard one. $\diamondsuit$

Remark (Positive homogeneity excludes directional convexity) Combining Theorems 2.5 and 2.6 shows that every convex payoff function h that is both positively homogeneous and directionally convex must be linear. In fact, the additional assumption of convexity can be dropped as is shown by the following direct argument. For $n = 2$ and $x_1, x_2 > 0$, positive homogeneity yields

$$\frac{h(x_1 + t, x_2) - h(x_1, x_2)}{t} = \frac{h(x_1/x_2 + (t/x_2), 1) - h(x_1/x_2, 1)}{t/x_2}.$$

Sending $t \downarrow 0$ yields

$$h^+_{x_1}(x_1, x_2) = h^+_{x_1}(x_1/x_2, 1).$$

Since h is directionally convex, the left-hand side is increasing and the right-hand side is decreasing in x_2. Thus, $h^+_{x_1}(x_1, x_2)$ is independent of x_2, and the continuity and positive homogeneity of h imply that

$$h(x_1, x_2) = h(0, x_2) + \int_0^{x_1} h^+_{x_1}(u, x_2)\, du = h(0, x_2) + h(x_1, 0) + h(0, 0) = x_2 h(0, 1) + x_1 h(1, 0),$$

i.e., h is linear. For $n > 2$ one can use induction and Lemma 3.4 below. ♢

Remark (Recalibration) Our approach assumes that the local volatility model is set up at time $t = 0$ and not altered in the sequel. In practice, market models sometimes undergo a recalibration process during which model parameters are refitted at intermediate times. In our case, this would mean that the volatility function $\sigma(t, x)$ would become dependent in a highly nontrivial way on the prices of the underlying and all plain vanilla options observable at time t. Recalibration will therefore considerably complicate the situation. It is interesting to note that recalibration may in fact destabilize pricing and hedging of derivatives; see Bühler [2, 3] for a recent analysis of parameter recalibration in stochastic volatility models. ♢

Remark (Alternate conditions on σ and h) Instead of Assumption 2.1 we could also work with the following set of conditions:

(a') $\sigma : [0, T] \times (0, \infty) \to (0, \infty)$ is continuous, bounded, locally Hölder continuous in (t, x), and continuously differentiable with respect to x.

(b') $\partial\sigma(t, x)/\partial x$ is locally Hölder continuous in (t, x), and there exists some constant $A > 0$ such that

$$\Big|\frac{\partial}{\partial x}\sigma(t, x)\Big| \le A\frac{\log(1 + x)}{1 + x}.$$

Suppose moreover that the payoff function $h(x_1, \dots, x_n)$ is continuously differentiable and its partial derivatives grow at most logarithmically. If these conditions are satisfied, then variants of Proposition 2.2 and Theorem 2.6 hold. In addition, the option's Delta $v_x(t, \cdot, x)$ grows at most logarithmically in x; see Stadje [21]. ♢

3 Proofs

We recall that we work under the conditions of Assumption 2.1. The following lemma relies on standard facts on parabolic partial differential equations.

Lemma 3.1 *Let* $h : [0, \infty) \to (0, \infty)$ *be a payoff function. Then* $v(t, x) := E[\, h(S_T) \,|\, S_t = x \,]$ *belongs to* $C([0, T] \times [0, \infty)) \cap C^{1,2}([0, T) \times (0, \infty))$*, satisfies a polynomial growth condition uniformly in* $t \in [0, T]$*, and solves the Cauchy problem*

$$v_t(t, x) + \frac{1}{2}\sigma^2(t, x)x^2 v_{xx}(t, x) = 0 \text{ in } (0, T) \times (0, \infty), \text{ and } v(T, x) = h(x).$$

Proof: According to [17, Theorem A.14], the above Cauchy problem has a unique solution $v \in C([0, T] \times [0, \infty)) \cap C^{1,2}([0, T) \times (0, \infty))$ with polynomial growth. Thus, the assertion follows from [18, Theorem 5.7.6]. □

Proof of Proposition 2.2: The idea is to apply Lemma 3.1 and backward induction. To this end, let us introduce the dummy variables $x_1, \dots, x_{n-1} \in (0, \infty)$ and the function $h_1(x) := h(x_1, \dots, x_{n-1}, x)$. For $t_{n-1} \le t < t_n$, we can directly apply Lemma 3.1 and conclude that $v(t, x_1, \dots, x_{n-1}, x) = E[\, h_1(S_{t_n}) \,|\, S_t = x \,]$ satisfies the regularity conditions and the PDE in question.

For the case $t_{n-2} \le t < t_{n-1}$, we define

$$h_2(x) := E[\, h(x_1, \dots, x_{n-2}, x, S_{t_n}) \,|\, S_{t_{n-1}} = x \,].$$

Then we have by the Markov property of S that

$$\begin{aligned} v(t, x_1, \dots, x_{n-2}, x) &= E[\, h(x_1, \dots, x_{n-2}, S_{t_{n-1}}, S_{t_n}) \,|\, S_t = x \,] \\ &= E[\, h_2(S_{t_{n-1}}) \,|\, S_t = x \,]. \end{aligned}$$

Hence, the assertion will follow for $t_{n-2} \le t < t_{n-1}$ by Lemma 3.1 if we can show that h_2 is again a payoff function. Let us show first the polynomial growth condition. Due to the polynomial growth condition for h, there are constants $C, p \ge 1$ such that

$$h(x_1, \dots, x_{n-2}, x, S_{t_n}) \le C(1 + x_1^p + \dots + x_{n-2}^p + x^p + S_{t_n}^p). \tag{15}$$

Thus, according to Lemma 3.1, h_2 also satisfies the polynomial growth condition. As to the proof of the continuity of h_2, let $\widetilde{h}(y, z) := h(x_1, \dots, x_{n-2}, y, z)$. Then

$$\begin{aligned} |h_2(x) - h_2(y)| \le{}& \big|E[\, \widetilde{h}(x, S_{t_n}) \,|\, S_{t_{n-1}} = x \,] - E[\, \widetilde{h}(x, S_{t_n}) \,|\, S_{t_{n-1}} = y \,]\big| \\ &+ \big|E[\, \widetilde{h}(x, S_{t_n}) - \widetilde{h}(y, S_{t_n}) \,|\, S_{t_{n-1}} = y \,]\big|. \end{aligned}$$

As $x \to y$, the first difference on the right tends to zero by Lemma 3.1. For the second difference, we can use (15) and dominated convergence (see the Remark following Assumption 2.1). Thus, h_2 is also continuous. Backward induction on n concludes the proof of the regularity assertions of v and of (9).

We now prove Itô's formula (11). Due to the established regularity properties of v, we may apply Föllmer's pathwise Itô formula [7] for $t_k < s < t < t_{k+1}$:

$$v(t,X) = v(s,X) + \int_s^t v_x(r,X)\,dX_r + \frac{1}{2}\int_s^t v_{xx}(r,X)\,d\langle X\rangle_r + \int_s^t v_t(r,X)\,dr.$$

Since v is continuous and (10) holds, we may pass to the limit as $s \downarrow t_k$ and $t \uparrow t_{k+1}$. Summing over k then yields (11). □

Proof of Corollary 2.4: Due to the two Propositions 2.2 and 2.3, we only have to prove the convexity of $x \mapsto v(t,\cdot,x)$. But this follows immediately from the fact that the path of a geometric Brownian motion with time-dependent volatility $\sigma(t)$ and $S_0 = x$ is given by

$$S_t = x \exp\Big(\int_0^t \sigma(s)\,dW_s - \frac{1}{2}\int_0^t \sigma^2(s)\,ds\Big), \qquad t \geq 0,$$

and thus is affine in its starting point x. □

Now we will prepare for the proof of Theorem 2.5. We will denote by S^γ_t the price process in a Black & Scholes model with constant volatility $\gamma > 0$. More precisely,

$$S^\gamma_t = S_0 \exp\big(\gamma W_t - \frac{1}{2}\gamma^2 t\big).$$

We will write v^γ for the value function corresponding to $H(S^\gamma)$. Note that if h is positively homogeneous, then so will be $v^\gamma(t,\cdot)$.

Lemma 3.2 *For $S^\gamma_0 = 1$, the function $\gamma \mapsto E[\,H(S^\gamma)\,]$ is strictly increasing if h satisfies the assumtions of Theorem 2.5.*

Proof: We will proceed by induction on n. For $n = 1$, the convex function h can be represented as

$$h(x) = h(0) + h'(0+)x + \int_{(0,\infty)} (x-K)^+\,\mu(dK) \tag{16}$$

for a positive Radon measure μ on $(0,\infty)$; see, e.g., Example 1.24 in [10]. This measure μ must be nonzero since h is not linear. Fubini's theorem thus yields that for all $t, \gamma > 0$

$$E[\,h(S^\gamma_t)\,] = h(0) + h'(0+) + \int_{(0,\infty)} E[\,(S^\gamma_t - K)^+\,]\,\mu(dK).$$

It follows from the standard Black & Scholes formula that $\gamma \mapsto E[\,(S^\gamma_t - K)^+\,]$ is strictly increasing, and this proves our claim for $n = 1$. Suppose now that our claim has already been established for $n \geq 1$ and that $h(x_1,\dots,x_{n+1})$ is a convex payoff function, which is not linear. If $(x_2,\dots,x_{n+1}) \mapsto h(x_1,x_2,\dots,x_{n+1})$ is linear for $x_1 = 1$ and hence for all $x_1 > 0$, then we are back in the situation where $n = 1$. Otherwise, the induction hypothesis and the Markov property of S^γ yield that

$$\gamma \longmapsto E[\,h(1, S^\gamma_{t_2-t_1},\dots,S^\gamma_{t_{n+1}-t_1})\,] = \frac{1}{x} v^\gamma(t_1,x,x)$$

is strictly increasing for all $x > 0$. Thus, we get that for $\gamma < \kappa$

$$E[\,H(S^\gamma)\,] = E[\,v^\gamma(t_1, S^\gamma_{t_1}, S^\gamma_{t_1})\,] = v^\gamma(t_1,1,1) < v^\kappa(t_1,1,1) = E[\,H(S^\kappa)\,].$$

This concludes the proof of our lemma. □

Proof of Theorem 2.5: We will show that $v(0,x) = E_x[\,H(S)\,]$ is not convex, where the subscript x indicates that we are considering the model with $S_0 = x$. To this end, it is enough to show that

$$\liminf_{x\downarrow 0} \frac{v(0,x)}{x} > \limsup_{y\uparrow\infty} \frac{v(0,y)}{y}. \tag{17}$$

Let $\overline{\sigma}(t,x) := C \vee \sigma(t,x)$ and consider the solution X of the SDE

$$dX_t = \overline{\sigma}(t,X_t)X_t\,dW_t, \qquad X_0 = S_0.$$

Due to the pathwise uniqueness of our SDEs, the sample paths of S and X coincide up to

$$\tau := \inf\{t \geq 0 \,|\, \sigma(t,S_t) \leq C\}.$$

Moreover, C underestimates the 'market volatility' of X, i.e., $C^2X_t^2\,dt \leq \overline{\sigma}(t,X_t)^2\,dt = d\langle X\rangle_t$, and Corollary 2.4 yields that

$$v^C(0,X_0) + \int_0^T v_x^C(s,X)\,dX_s \leq H(X).$$

Since σ is bounded from above, X is a true martingale due to Novikov's theorem. Moreover, it follows from the positive homogeneity and continuity of h that $v^\Sigma(t,\cdot,x)$ has at most linear growth in x. Hence $v_x^\Sigma(s,X)$ is bounded for a.e. s, and it follows that $\int v_x^C(s,X)\,dX_s$ is a true martingale. Taking expectations thus yields

$$E_x[\,H(X)\,] \geq v^C(0,x) = xv^C(0,1).$$

Thus,

$$\begin{aligned} v(0,x) = E_x[\,H(S)\,] &\geq E_x[\,H(S);\,\tau \geq T\,] = E_x[\,H(X);\,\tau \geq T\,] \\ &\geq xv^C(0,1) - E_x[\,H(X);\,\tau < T\,]. \end{aligned}$$

We will show below that

$$\frac{1}{x}\cdot E_x[\,H(X);\,\tau < T\,] \longrightarrow 0 \qquad \text{as } x \downarrow 0. \tag{18}$$

This will then yield

$$\liminf_{x\downarrow 0} \frac{v(0,x)}{x} \geq v^C(0,1).$$

Similarly, one shows that

$$\limsup_{y\uparrow\infty} \frac{v(0,y)}{y} \leq v^c(0,1),$$

and this will then complete the proof of (17) when combined with Lemma 3.2.

To prove (18), note first that there is a constant $k \geq 0$ such that $H(X) \leq k\max_t X_t$, due to the continuity and positive homogeneity of h. Writing $X_t = xM_t$, where the martingale M is the Doleans-Dade exponential $\mathcal{E}(\int_0 \overline{\sigma}(t,X_t)\,dW_t)_t$, we see that

$$\frac{1}{x}\cdot E_x[\,H(X);\,\tau < T\,] \leq kE_x[\,\max_{t\leq T} M_t;\,\tau < T\,] \leq kE_x[\,\max_{t\leq T} M_t^2\,]^{1/2}P_x[\,\tau < T\,]^{1/2}.$$

Next, by Doob's inequality,

$$E_x[\max_{t\le T} M_t^2] \le 2E_x[M_T^2] \le 2e^{\Sigma T},$$

where Σ is an upper bound for σ. Furthermore, S is a true martingale, and so

$$x = E_x[S_{\tau\wedge T}] \ge E_x[S_{\tau\wedge T};\, \tau < T] \ge x_0 P_x[\tau < T].$$

Thus, $P_x[\tau < T] \to 0$ as $x \downarrow 0$; and (18) follows.

It remains to construct a path X along which the Delta hedging strategy is not robust. Due to the the non-convexity of $v(0,\cdot)$, there is some $x > 0$ such that $v_{xx}(0,x) < 0$. Now let X^1 be a typical path of a geometric Brownian motion with a volatility γ, which is a strict lower bound for $\sigma(\cdot)$, and suppose that $X_0^1 = x$. The continuity of v_{xx} implies that

$$\tau := \inf\{t \ge 0 \,|\, v_{xx}(t, X_t^1) \ge 0\} \wedge \frac{t_1}{2}$$

is strictly positive. Let furthermore X^2 be a typical path of the solution to the SDE $dS_t = \sigma(t,S_t)S_t\,dW_t$ such that $X_\tau^2 = X_\tau^1$. We claim that $X_t := X_t^1 \mathrm{I}_{\{t\le\tau\}} + X_t^2 \mathrm{I}_{\{t>\tau\}}$ does the job. First note that $d\langle X\rangle_t = \gamma^2 X_t^2\,dt < \sigma^2(t,X_t)X^2\,dt$ for $t \le \tau$ and $d\langle X\rangle_t = \sigma^2(t,X_t)X^2\,dt$ for $t > \tau$, so that σ overestimates the volatility of X. Moreover, we have

$$v(0,S_0) + \int_0^\tau v_x(t,S_t)\,dS_t = v(\tau,S_\tau) \qquad \text{and} \qquad v(\tau,X_\tau) + \int_\tau^T v_x(\tau,X)\,dX_t = H(X).$$

These Itô integrals are all well-defined, due to the martingale property of $v(t,S)$ and the martingale representation theorem for S. By using $v_{xx}(t,X_t) < 0$ for $t < \tau$ and arguing as in the proof of Proposition 2.3, we obtain

$$v(0,X_0) + \int_0^\tau v_x(t,X_t)\,dX_t < v(\tau,X_\tau)$$

and thus

$$v(0,X_0) + \int_0^T v_x(t,X)\,dX_t < H(X),$$

i.e., the Delta hedging strategy is not robust along the path X. □

Let us now turn to the proof of Theorem 2.6. We prepare its proof with the following two auxiliary results.

Proposition 3.3 *Suppose that $h(x_1,\dots,x_n)$ is a directionally convex payoff function and $0 \le t < T$ are given. Then the function*

$$u(x_1,\dots,x_n) := E[\,h(x_1,\dots,x_{n-1},S_T) \,|\, S_t = x_n\,]$$

is also directionally convex.

Proof: Due to Proposition 2.2 and the Remark following Theorem 2.6, we have to show that

$$x_j \longmapsto u(x_1,\dots,x_i+\varepsilon,\dots,x_n) - u(x_1,\dots,x_i,\dots,x_n) \tag{19}$$

is increasing for all $\varepsilon > 0$ and $i,j \in \{1,\dots,n\}$. This property is clear for $i,j < n$. To prove it for the remaining cases, let $S_T^{t,x}$ denote the solution of $dS_r = \sigma(r,S_r)S_r\,dW_r$ starting at time t in $S_t^{t,x} = x$. Due to a standard comparison result, we have $S_T^{t,y} \geq S_T^{t,x}$ P-a.s. for $y \geq x$; see, e.g., Proposition 5.2.18 in [18]. Hence, (19) is increasing for $i < n$ and $j = n$ or for $i = n$ and $j < n$. For $i = j = n$, we have to show the convexity of u in its final argument. For the case that h growths at most linearly in x_n, this follows from Theorem 5.2 by El Karoui et al. [6]. The general case then can be deduced by representing the convex function $x_n \mapsto h(x_1,\dots,x_n)$ as in (16). □

Lemma 3.4 *If $h(x_1,\dots,x_{n+1})$ is directionally convex, then so is the contraction*

$$g(x_1,\dots,x_{n-1},x_n) := h(x_1,\dots,x_{n-1},x_n,x_n).$$

Proof: Due to the Remark following Theorem 2.6, we have to show that

$$x_j \longmapsto g(x_1,\dots,x_i+\varepsilon,\dots,x_n) - g(x_1,\dots,x_i,\dots,x_n)$$

is increasing for all $\varepsilon > 0$ and $i,j \in \{1,\dots,n\}$. This property is clear for $i < n$. If $i = n$, then

$$\begin{aligned} &g(x_1,\dots,x_{n-1},x_n+\varepsilon) - g(x_1,\dots,x_{n-1},x_n) \\ &\quad = \big[h(x_1,\dots,x_n+\varepsilon,x_n+\varepsilon) - h(x_1,\dots,x_n+\varepsilon,x_n)\big] \\ &\qquad + \big[h(x_1,\dots,x_n+\varepsilon,x_n) - h(x_1,\dots,x_n,x_n)\big] \end{aligned}$$

Clearly, either of the two terms in brackets is increasing in any of the variables $x_1,\dots,x_n$, and the result follows. □

Proof of Theorem 2.6: Take some $t_{n+1} > t_n$. We will prove our assertion for $t_k \leq t < t_{k+1}$ by backward induction on k. There is nothing to show if $k = n$. So let us suppose that the directional convexity of $v(t_{k+1},x_1,\dots,x_{k+2})$ with respect to $x_1,\dots,x_{k+2}$ has already been established and that $t_k \leq t < t_{k+1}$. Using the Markov property of S, we see that

$$\begin{aligned} v(t,x_1,\dots,x_k,x) &= E\big[\,v(t_{k+1},x_1,\dots,x_k,S_{t_{k+1}},S_{t_{k+1}})\,\big|\,S_t = x\,\big] \\ &= E\big[\,g(x_1,\dots,x_k,S_{t_{k+1}})\,\big|\,S_t = x\,\big], \end{aligned}$$

where $g(x_1,\dots,x_k,x_{k+1}) := v(t_{k+1},x_1,\dots,x_k,x_{k+1},x_{k+1})$. By Lemma 3.4 and the induction hypothesis, the function g is directionally convex. Hence, the assertion follows from Proposition 3.3. □

Acknowledgement: It is a pleasure to thank Hans Bühler and Stephan Sturm for discussions. We also thank Antje Schulz for comments.

References

[1] Bergenthum, J. Rüschendorf, L. *Comparison of option prices in semimartingale models.* Preprint, Universität Freiburg (2004).

[2] Bühler, H. *Consistent variance swap models.* Preprint, TU Berlin, 2004.

[3] Bühler, H. *Volatility markets. Consistent modeling, hedging and practical implementation.* PhD thesis, TU Berlin (2005).

[4] Dupire, B. *Pricing and hedging with smiles.* Mathematics of derivative securities (Cambridge, 1995), 103–111, Publ. Newton Inst., 15, Cambridge Univ. Press, Cambridge, 1997.

[5] Ekström, E., Janson, S., Tysk, J. *Superreplication of options on several underlying assets.* J. Appl. Probab. 42 (2005), no. 1, 27–38.

[6] El Karoui, N., Jeanblanc-Picqué, M., Shreve, S. *Robustness of the Black and Scholes formula.* Math. Finance 8, no. 2, 93–126 (1998).

[7] Föllmer, H., *Calcul d'Itô sans probabilités.* Seminaire de probabilites XV, Univ. Strasbourg 1979/80, Lect. Notes Math. 850 (1981), 143-150.

[8] Föllmer, H. *Probabilistic Aspects of Options.* Rolf Nevanlinna Institute Reports B6, Helsinki (1991)

[9] Föllmer, H., *Probabilistic Aspects of Financial Risk.* Plenary Lecture at the Third European Congress of Mathematics. Proceedings of the European Congress of Mathematics, Barcelona 2000, Birkhäuser, Basel, 2001.

[10] Föllmer, H., Schied, A. *Stochastic Finance: An Introduction in Discrete Time.* Berlin: de Gruyter Studies in Mathematics 27 (2002). Second edition (2004).

[11] Gushchin, A., Mordecki, E. *Bounds on option prices for semimartingale market models.* Proc. Steklov Inst. Math. 237, 73113 (2002).

[12] Gyöngy, I. *Mimicking the one-dimensional marginal distributions of processes having an Itô differential.* Probab. Theory Relat. Fields 71 (1986), no. 4, 501–516.

[13] Hajek B. *Mean Stochastic Comparison of Diffusions.* Z. Wahrscheinlichkeitstheorie verw. Gebiete, 68, 315-329 (1985).

[14] Hardy, G. H., Littlewood, J. E., Pólya, G., *Inequalities.* 2^{nd} edition, Cambridge University Press, Cambridge, 1952.

[15] Hobson, D. *Volatility misspecification, option pricing and superreplication via coupling.* Ann. Appl. Probab. 8 (1998), no. 1, 193–205.

[16] Janson, S., Tysk, J. *Volatility time and properties of option prices.* Ann. Appl. Probab. 13 (2003), no. 3, 890–913.

[17] Janson, S., Tysk, J. *Preservation of convexity to solutions of parabolic equations.* J. Differential Equations 206, 182-226 (2004)

[18] Karatzas, I., Shreve, S. *Brownian Motion and Stochastic Calculus*, Springer, Berlin (1991).

[19] Lyons, T. *Uncertain volatility and the risk-free synthesis of derivatives.* Applied Math. Finance 2, 117-133 (1995).

[20] Schied, A. *Optimal investments for robust utility functionals in complete market models.* Math. Oper. Research. 30, No. 3, 750-764 (2005).

[21] Stadje, M. *Convexity of option prices in a local volatility model.* Diploma thesis, TU Berlin, 2005.

[22] Wright, E. M. *An inequality for convex functions.* Amer. Math. Monthly 61, (1954). 620–622.