Robust maximization of consumption with logarithmic utility

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Abstract— **We analyze the stochastic control approach to the dynamic maximization of the robust utility of consumption and investment. The robust utility functionals are defined in terms of logarithmic utility and a dynamically consistent convex risk measure. The underlying market is modeled by a diffusion process whose coefficients are driven by an external stochastic factor process. Our main results give conditions on the minimal penalty function of the robust utility functional under which the value function of our problem can be identified with the unique classical solution of a quasilinear PDE within a class of functions satisfying certain growth conditions.**

I. INTRODUCTION

One of the fundamental problems in mathematical finance and mathematical economics is the construction of investment strategies that maximize the utility functional of a risk-averse investor. In the majority of the corresponding literature, the optimality criterion is based on a classical expected utility functional of von Neumann-Morgenstern form, which requires the choice of a single probabilistic model $\mathbb{P}$. In reality, however, the choice of $\mathbb{P}$ is often subject to model uncertainty. Schmeidler [**?**] and Gilboa and Schmeidler [10] therefore proposed the use of *robust utility functionals* of the form

$$X \longmapsto \inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,], \tag{1}$$

where $\mathcal{Q}$ is a set of prior probability measures. In analogy to the move from coherent to convex risk measures, Maccheroni et al. [16] recently suggested to model investor's preferences by robust utility functionals of the form

$$X \longmapsto \inf_{Q} \big(E_Q[\,U(X)\,] + \gamma(Q)\big), \tag{2}$$

where γ is a penalty function defined on the set of all possible probabilistic models.

Optimal investment problems for robust utility functionals (1) were considered, among others, by Talay and Zheng [24], Quenez [18], Schied [19], Burgert and Rüschendorf [3], Schied and Wu [23], Föllmer and Gundel [8], and the authors [13]. For the generalized utility functionals of type (2), the most popular choice for the penalty function has so far been the entropic penalty function $\gamma(Q) = kH(Q|\mathbb{P})$ for a constant $k > 0$ and a reference probability measure $\mathbb{P}$; see, e.g., Hansen and Sargent [12] and Bordigoni et al. [2]. The duality theory for the optimal investment problem with a general penalty function γ was developed by Schied [21]. It was extended by Wittmüss [25] to the consumption-investment problem with random endowment.

In this paper, we propose a stochastic control approach to the dynamic maximization of robust utility functionals of the form (2). The penalty function γ will be defined in a Brownian setting and, apart from certain basic requirements such as time consistency, has a rather general form. In particular, we will go beyond the very particular situation of entropic penalties and include the 'coherent' setting (1) as a special case. Our setting will involve logarithmic utility $U(x) = \log x$ and an incomplete financial market model, whose volatility, interest rate process, and trend are driven by an external stochastic factor process. In this setting, the control approach to the optimization of the terminal wealth was developed by the authors in a previous paper [14]. We now extend these results by also allowing for intertemporal consumption.

Our goal consists in characterizing the value function and the optimal investment strategy via the solution of a quasilinear Hamilton-Jacobi-Bellman PDE. As a byproduct, we also obtain a formula for the least-favorable martingale measure in the sense of Föllmer and Gundel [8]. In contrast to earlier approaches such as [24], we avoid the use of viscosity solutions and concentrate our effort on obtaining strong regularity results, which allow us to identify the value function as a unique classical solution of the PDE in question. Regularity of solutions is important because it facilitates the use of standard numerical methods for solving the PDE, and we will use such methods in illustrating some interesting qualitative properties of the optimal strategy.

Our method consists in combining the duality results from [21] and [25] with a PDE approach to the dual problem of determining optimal martingale measures. This technique has already been applied successfully by Castañeda-Leyva and Hernández-Hernández [4], [5] to the maximization of von Neumann-Morgenstern expected utility and by the authors [13] and Schied [22] in the maximization of 'coherent' robust utility functionals of the form (1). The approach has also been applied successfully to the maximization of the robust utility (2) of the terminal wealth; see [14]. In this paper, we extend the results from [14] to the case of combined intertemporal consumption and investment.

In the next section we describe the set-up of our problem and state the theorems for our main findings. Their proofs can be obtained by appropriately adapting the proofs in [13]. Specifically, one has to replace the duality results from [21] by those from [25]. In setting up the control approach to the

*Supported by Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

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dual problem and in identifying the optimal consumption-investment strategy, one can follow the arguments in [22].

II. Statement of main results

We consider a financial market model with a locally riskless money market account

$$dS_t^0 = S_t^0 r(Y_t)\,dt$$

and a risky asset defined under a reference measure $\mathbb{P}$ through the SDE

$$dS_t = S_t b(Y_t)\,dt + S_t\sigma(Y_t)\,dW_t^1.$$

Here W^1 is a standard $\mathbb{P}$-Brownian motion and Y denotes an external economic factor process modeled by the SDE

$$dY_t = g(Y_t)\,dt + \rho\,dW_t^1 + \overline{\rho}\,dW_t^2, \tag{3}$$

where $\rho \in [-1,1]$ is some correlation factor, $\overline{\rho} := \sqrt{1-\rho^2}$, and W^2 is a standard $\mathbb{P}$-Brownian motion, which is independent of W^1 under $\mathbb{P}$. We suppose that the economic factor cannot be traded directly so that the market model will typically be incomplete.

We assume that $g(\cdot)$ is in $C^2(\mathbb{R})$ with derivative $g' \in C_b^1(\mathbb{R})$, and $r(\cdot)$, $b(\cdot)$, and $\sigma(\cdot)$ belong to $C_b^2(\mathbb{R})$, where $C_b^k(\mathbb{R})$ denotes the class of bounded functions with bounded derivatives up to order k. The 'market price of risk' is defined via the function

$$\theta(y) := \frac{b(y)-r(y)}{\sigma(y)},$$

and we will assume that $\sigma(\cdot) \geq \sigma_0 > 0$ for some constant σ_0. The assumption of time-independent coefficients is for notational convenience only and can easily be relaxed.

In most economic situations, investors typically face *model uncertainty* in the sense that the dynamics of the relevant quantities are not precisely known. One common approach to coping with model uncertainty is to allow in principle all probability models corresponding to probability measures $Q \ll \mathbb{P}$ and to penalize each such model with a penalty $\gamma(Q)$. To define $\gamma(Q)$, we assume henceforth that everything is modeled on the canonical path space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ of $W = (W^1, W^2)$. Then every probability measure $Q \ll \mathbb{P}$ admits a progressively measurable process $\eta = (\eta_1, \eta_2)$ such that

$$\frac{dQ}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta_{1t}\,dW_t^1 + \int_0 \eta_{2t}\,dW_t^2\Big)_T \quad Q-\text{a.s.},$$

where $\mathcal{E}(M)_t = \exp(M_t - \langle M\rangle_t/2)$ denotes the Doleans-Dade exponential of a continuous semimartingale M. Such a measure Q will receive a penalty

$$\gamma(Q) := E_Q\Big[\int_0^T h(\eta_t)\,dt\Big], \tag{4}$$

where $h : \mathbb{R}^2 \to [0,\infty]$ is convex and lower semicontinuous. For simplicity, we will suppose $h(0) = 0$ so that $\gamma(\mathbb{P}) = 0$. We will also assume that h is continuously differentiable on its effective domain $\operatorname{dom} h := \{\eta \in \mathbb{R}^2 \,|\, h(\eta) < \infty\}$ and satisfies the coercivity condition

$$h(x) \geq \kappa_1|x|^2 - \kappa_2 \qquad \text{for constants } \kappa_1,\ \kappa_2 > 0. \tag{5}$$

The choice $h(x) = |x|^2/2$ corresponds to the entropic penalty function considered in Hansen and Sargent [12] and Bordigoni et al. [2]; see Remark 2.6 below. Again, our assumption that h does not depend on time is for notational convenience only.

Let $\mathcal{A}$ denote the set of all pairs (c,π) of progressively measurable processes such that $c \geq 0$, $\int_0^T c_s\,ds < \infty$, and $\int_0^T \pi_s^2\,ds < \infty$ $\mathbb{P}$-a.s. For $(c,\pi) \in \mathcal{A}$ and $x > 0$, we define $X^{x,c,\pi}$ as the unique solution of the linear SDE

$$dX_t^{x,c,\pi} = \frac{X_s^{x,c,\pi}(1-\pi_s)}{S_s^0}\,dS_s^0 + \frac{X_s^{x,c,\pi}\pi_s}{S_s}\,dS_s - c_s\,ds$$

with initial value $X_0^{x,c,\pi} = x$. The process $X^{x,c,\pi}$ thus describes the evolution of the wealth process of an investor with initial endowment $X_0^{x,c,\pi} = x > 0$ investing the fraction π_s of the current wealth into the risky asset at time $s \in [0,T]$. By $\mathcal{A}(x)$ we denote the subclass of all $(c,\pi) \in \mathcal{A}$ that are admissible in the sense that $X_t^{x,c,\pi} \geq 0$ $\mathbb{P}$-a.s. for all t.

The objective of the investor is to maximize

$$\inf_{Q \ll \mathbb{P}} \Big(E_Q\Big[\int_0^T U(c_t)\,dt + U(X_T^{x,c,\pi})\Big] + \gamma(Q)\Big) \tag{6}$$

over $(c,\pi) \in \mathcal{A}(x)$. The utility function $U : (0,\infty) \to \mathbb{R}$ will be specified in the sequel as a HARA utility function with risk aversion parameter $\alpha = 0$, i.e.,

$$U(x) = \log x. \tag{7}$$

Our goal is to characterize the value function

$$u(x) := \sup_{\pi \in \mathcal{A}} \inf_{Q \ll \mathbb{P}} \Big(E_Q\Big[\int_0^T \log c_t\,dt + \log X_T^{x,c,\pi}\Big] + \gamma(Q)\Big)$$

of the robust utility maximization problem (6) in terms of the solution v of the quasi-linear parabolic initial value problem

$$\begin{cases} v_t = \frac{1}{2}v_{yy} + \phi(v_y) + gv_y + (1+t)r \\ v(0,\cdot) = 0, \end{cases} \tag{8}$$

where the nonlinearity $\phi(v_y) = \phi(t,y,v_y(t,y))$ is given by

$$\phi(t,y,z) := \psi(t,y,(\rho,\overline{\rho})z) \qquad y,z \in \mathbb{R}.$$

for the function

$$\psi(t,y,x) := \inf_{\eta \in \mathbb{R}^2} \Big\{\eta \cdot x + \frac{1}{2}(1+t)(\eta_1 + \theta(y))^2 + h(\eta)\Big\},$$

with $y \in \mathbb{R},\ x \in \mathbb{R}^2$. Here, $\eta \cdot x$ denotes the inner product of η and x. The easy case is the one in which the effective domain of h is compact:

Theorem 2.1: Suppose that $\operatorname{dom} h$ is compact. Then the value function u of the robust utility maximization problem satisfies

$$u(x) = (1+T)\log x + v(T, Y_0),$$

where $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the unique classical solution to (8) within the class of functions in $C^{1,2}((0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ satisfying a polynomial growth condition.

Suppose furthermore that $\eta^* : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\eta^*(t,y)$ belongs to the supergradient of the concave function $x \mapsto \psi(t,y,x)$ at $x=(\rho,\overline{\rho})v_y(t,y)$. Then an optimal strategy $(\widehat{c},\widehat{\pi}) \in \mathcal{A}(x)$ for the robust problem can be obtained by letting

$$\widehat{\pi}_t = \frac{\eta_1^*(T-t,Y_t)+\theta(Y_t)}{\sigma(Y_t)}$$

and by consuming at a rate proportional to the current total wealth $X_t^{x,\widehat{c},\widehat{\pi}}$:

$$\widehat{c}_t = \frac{1}{1+T-t} X_t^{x,\widehat{c},\widehat{\pi}}.$$

Moreover, by defining a measure $\widehat{Q} \sim \mathbb{P}$ via

$$\frac{d\widehat{Q}}{d\mathbb{P}} = \mathcal{E}\Big(\int_0 \eta^*(T-t,Y_t)\,dW_t\Big)_T, \tag{9}$$

we obtain a saddlepoint $(\widehat{\pi},\widehat{Q})$ for the maximin problem (6).

The regularity of the value function obtained in the preceding theorem is important, because it facilitates the use of standard numerical methods for solving the PDE (8).

Remark 2.2: The proof of Theorem 2.1 shows that the probability measure P^* with density

$$\frac{dP^*}{d\mathbb{P}} = \mathcal{E}\Big(-\int_0 \theta(Y_s)\,dW_s^1 + \int_0 \eta_2^*(T-s,Y_s)\,dW_s^2\Big)_T$$

is a least favorable martingale measure in the sense of Föllmer and Gundel [8]. This will also be true in the setting of Theorems 2.3 and 2.5.

The problem becomes more difficult when dom h is noncompact, because then we can no longer apply standard theorems on the existence of classical solutions to (8). Other difficulties appear when dom h is not only noncompact but also unbounded. For instance, we may have $\gamma(Q)<\infty$ even if Q is not equivalent but merely absolutely continuous with respect to $\mathbb{P}$, and this leads to difficulties when one tries to work directly on the primal problem. Moreover, since the optimal η^* takes values in the unbounded set dom h, one needs an additional argument to ensure that the stochastic exponential in (9) is a true martingale and hence defines a probability measure $\widehat{Q} \ll \mathbb{P}$. Our strategy to get the necessary integrability of the process $\eta_1^*(T-t,Y_t)$ is to use qualitative properties of solutions v to (8) as to control the growth of the gradient v_y. In doing so, we have to eliminate the possible competition between the linear term gv_y and the nonlinear term $\phi(v_y)$ by imposing a growth condition on ϕ.

Theorem 2.3: Suppose that g is bounded and that there exists some $\varepsilon>0$ such that

$$\liminf_{|p|\to\infty} \Big|\frac{\phi(t,y,p)}{p}\Big| \ge \varepsilon + |g(y)|. \tag{10}$$

Then the value function u of the robust utility maximization problem satisfies $u(x)=(1+T)\log x + v(T,Y_0)$ where v is the unique classical solution of (8) within the class of functions in $C^{1,2}((0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$ with bounded gradient v_y. Under these conditions, also the conclusions on the optimal strategy $(\widehat{c},\widehat{\pi})$ and the measure $\widehat{Q}$ in Theorem 2.1 remain true.

The most interesting case is the one in which both dom h and the function g are unbounded. Here we need an additional condition on the shape of the function ψ. Note that g is unbounded if, e.g., Y is an Ornstein-Uhlenbeck process.

Definition 2.4: Let $f:\mathbb{R}^2\to\mathbb{R}$ be an upper semicontinuous concave function. We will say that f satisfies a *radial growth condition in direction* $x\in\mathbb{R}^2$ if there exist positive constants p_0 and C such that

$$\max\big\{|z| \,\big|\, z\in\partial f(px)\big\} \le C\big(1+|\partial_p^+ f(px)| \vee |\partial_p^- f(px)|\big),$$

for $p\in\mathbb{R}$, $|p|\ge p_0$, where $\partial f(px)$ denotes the supergradient of f in px and $\partial_p^+ f(px)$ and $\partial_p^- f(px)$ are the right-hand and left-hand derivatives of the concave function $p\mapsto f(px)$.

Note that if f is of the form $f(x)=f_0(|x|)$ for some convex increasing function f_0, then the radial growth condition is satisfied in any direction $x\neq 0$ with constant $C=1$.

Theorem 2.5: Suppose that $|\phi(t,y,p)/p|\to\infty$ as $|p|\to\infty$ and assume that $\psi(t,y,\cdot)$ satisfies a radial growth condition in direction $(\rho,\overline{\rho})$, uniformly in y and t. Then the value function u of the robust utility maximization problem satisfies $u(x)=(1+T)\log x+v(T,Y_0)$ where v is the unique classical solution of (8) within the class of polynomially growing functions in $C^{1,2}((0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$ whose gradient satisfies a growth condition of the form

$$\big|\partial_p^- \phi\big(y; v_y(t,y)\big)\big| \vee \big|\partial_p^+ \phi\big(y; v_y(t,y)\big)\big| \le C_1(1+|y|)$$

for some constant C_1. Under these conditions, also the conclusions on the optimal strategy $(\widehat{c},\widehat{\pi})$ and the measure $\widehat{Q}$ in Theorem 2.1 remain true.

Remark 2.6: For $q>0$, the choice $h(x)=\frac{1}{2q}|x|^2$ corresponds to the penalty function $\gamma(Q)=\frac{1}{q}H(Q|\mathbb{P})$, where

$$H(Q|\mathbb{P}) = \int \frac{dQ}{d\mathbb{P}} \log\frac{dQ}{d\mathbb{P}}\,d\mathbb{P} = \sup_{Y\in L^\infty}\big(E_Q[\,Y\,] - \log\mathbb{E}[\,e^Y\,]\big)$$

is the relative entropy of Q with respect to $\mathbb{P}$. Due to the classical duality formula

$$\log\mathbb{E}[\,e^X\,] = \sup_{Q\in\mathcal{Q}}\big(E_Q[\,X\,] - H(Q|\mathbb{P})\big),$$

the above choices correspond to the utility functional

$$\inf_{Q\ll\mathbb{P}}\big(E_Q[\,\log X\,]+\gamma(Q)\big) = -\frac{1}{q}\log\mathbb{E}\big[\,e^{-q\log X}\,\big].$$

Hence, as long as there is no consumption, the robust utility maximization problem (6) is equivalent to the maximization of the standard expected utility $\mathbb{E}[\,U(X_T^{x,c,\pi})\,]$ for the HARA utility function $U(x)=-x^{-q}$. With nontrivial consumption, however, such a reduction is no longer possible, and our problem can no longer be formulated exclusively in terms of standard expected utility. This situation problem is covered as a special case of Theorem 2.5. Indeed, the function ψ has the quadratic form

$$\psi(t,y,x) = -\frac{1}{2}\Big(\frac{(1+t)q}{1+q}(x_1+\theta(y))^2 + qx_2^2 - \theta(y)^2\Big),$$

and it is easily checked that it satisfies the radial growth condition in any direction. See Hansen and Sargent [12] and

Bordigoni et al. [2] for earlier studies of the problem of optimal consumption with entropic penalties.

III. ACKNOWLEDGMENTS

Acknowledgement: Part of this work was done while the first author visited INRIA Rocquencourt, its support trough the project Mathfi is greatly appreciated as well as its hospitality. The second author is grateful for support and hospitality during a research visit at the National Center for Theoretical Sciences at Tsing-Hua University and the Academia Sinica in Taipei, where part of the research for this paper was conducted. He is indebted to Shuenn-Jyi Sheu and Ching-Tang Wu for many stimulating discussions.

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